Asymptotic behavior for a singular diffusion equation with gradient absorption

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Abstract

We study the large time behavior of non-negative solutions to the singular diffusion equation with gradient absorption

$$\partial_t u - \Delta_p u + |\nabla u|^q = 0 \quad \text{in } (0,\infty)\times\mathbb{R}^N,$$

for $p_c := 2N/(N+1) < p < 2$ and $p/2 < q < q_* := p - N/(N+1)$. We prove that there exists a unique very singular solution of the equation, which has self-similar form and we show the convergence of general solutions with suitable initial data towards this unique very singular solution.

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1 Introduction and results

The aim of the present paper is to study the large time behavior of non-negative solutions to the following equation with singular diffusion and gradient absorption:

$$\partial_t u - \Delta_p u + |\nabla u|^q = 0, \quad (t,x) \in Q_\infty := (0,\infty) \times \mathbb{R}^N, \tag{1.1}$$

for $p_c := 2N/(N+1) < p < 2$ and $p/2 < q < q_* := p - N/(N+1)$. We consider only non-negative initial data

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^N, \tag{1.2}$$

under suitable decay and regularity assumptions that will be specified later. Equation (1.1) presents a competition between the effects of the two terms: one term of singular diffusion $\Delta_p u := \mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right)$, which in our case is supercritical (that is, $p > p_c = 2N/(N+1)$) in order to avoid extinction in finite time, and another term of nonlinear absorption depending on the gradient $|\nabla u|^q$. Due to this competition, interesting mathematical features appear in some ranges of exponents p and q.

The qualitative theory of (1.1) for general exponents p and q developed very recently; indeed, while there are many (even classical ones) papers on nonlinear diffusion equations with zero order absorption, covering almost all possible cases, the study of the gradient absorption proved to be much more involved and brought a bunch of very interesting mathematical phenomena, some of them having been the subject of intensive research in the last decade. As expected, the first results were obtained in the semilinear case $p = 2$, where the asymptotic behavior for $q > 1$ has been identified in a series of papers [4, 5, 7, 11, 12, 19, 20]. Finite time extinction was shown to take place for $q \in (0,1)$ [8, 9, 20] while the critical case $q = 1$, in spite of its apparent simplicity, is still far from being fully understood: only some large-time estimates are available [10] but no precise asymptotics. Passing to the p-Laplacian is a natural step, and for the slow-diffusion case $p > 2$, the exponent $q = p-1$ proved to have a very interesting critical effect, as an interface between absorption-dominated behavior and diffusion-dominated behavior [3, 28], while itself gives rise to a critical regularized sandpile-type behavior, as shown recently in [24]. A natural next step was then to pass to the study of the fast-diffusion case $1 < p < 2$, where the authors made important progress recently in understanding the decay rates and typical self-similar profiles [22, 23]. In particular, finite time extinction was shown to take place when (p,q) ranges in $(p_c, 2) \times (0, p/2)$ and in $(1, p_c) \times (0, \infty)$ while diffusion is likely to govern the large time dynamics when $(p,q) \in (p_c, 2) \times (q_*, \infty)$. The intermediate range $(p,q) \in (p_c, 2) \times (p/2, q_*)$ features a balance between the diffusion and absorption terms and is the focus of this paper.

From now on, we restrict ourselves to the following range of exponents:

$$p \in (p_c, 2) \quad \text{and} \quad q \in \left(\frac{p}{2}, q_*\right), \tag{1.3}$$

and we set

$$\alpha := \frac{p-q}{2q-p} > 0, \quad \beta := \frac{q-p+1}{2q-p} > 0 \quad \text{and} \quad \eta := \frac{1}{N(p-2)+p} > 0, \tag{1.4}$$

the positivity of η being a consequence of $p > p_c$. We also observe that, thanks to (1.3),

$$\alpha - N\beta = \frac{(N+1)(q_* - q)}{2q-p} > 0. \tag{1.5}$$

In order to state the main result concerning the large-time behavior, we recall a special category of solutions to (1.1), that are called *very singular solutions*. These are solutions to (1.1) with an initial trace at $t = 0$ more concentrated at the origin than a Dirac mass, thus justifying the name. The precise definition is given in Definition 4.1 at the beginning of Section 4.

The name *very singular solution* has been introduced in [14] for the heat equation with absorption of order zero. After this first paper, many other very singular solutions for diffusion equations with absorption terms were constructed, see [15, 27, 29, 31, 33, 35] and the references therein. For (1.1), we have established in [23, Theorem 1.1] the existence and uniqueness of such a very singular solution to (1.1), under the more restrictive hypothesis of radial symmetry and self-similarity. We recall this result for the reader's convenience as Theorem 4.2 below. For the moment, let us denote this unique radially symmetric, self-similar very singular solution by U with

$$U(t,x) := t^{-\alpha} f_U(xt^{-\beta}), \qquad (t,x) \in Q_\infty. \tag{1.6}$$

The main result about large time behavior is the following:

Theorem 1.1. *Let u_0 be a function such that*

$$u_0 \in L^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)\,, \quad u_0 \ge 0\,, \quad u_0 \not\equiv 0\,. \tag{1.7}$$

and

$$\lim_{|x|\to\infty} |x|^{\alpha/\beta} u_0(x) = 0. \tag{1.8}$$

Then, the following large time behavior holds true:

$$\lim_{t\to\infty} t^\alpha \|u(t) - U(t)\|_\infty = 0, \tag{1.9}$$

where U is the unique radially symmetric self-similar very singular solution to (1.1) *introduced in* (1.6).

In order to prove Theorem 1.1, several steps are needed, some of them being also very interesting by themselves. A very important element of the proof is identifying the possible limits as $t \to \infty$, that we can prove to be very singular solutions in the sense of Definition 4.1 by viscosity techniques. Thus, the circle will be closed by the following general uniqueness result.

Theorem 1.2. *There exists a unique very singular solution to* (1.1) *in the sense of Definition 4.1. In particular, this solution is radially symmetric and in self-similar form and it coincides with U.*

This theorem is an important extension of [23, Theorem 1.1], where the uniqueness of a very singular solution is established under the extra conditions of radial symmetry and self-similar form. An interesting by-product of Theorem 1.2 is a comparison principle for the elliptic equation

$$-\Delta_p v + |\nabla v|^q - \alpha v - \beta x \cdot \nabla v = 0, \quad x \in \mathbb{R}^N,$$

under suitable conditions as $|x| \to \infty$. For a precise form of the statement, we refer the reader to Theorem 4.15 below.

On the way to proving Theorem 1.2, we found out that a theory of the Cauchy problem associated to (1.1) with non-negative and bounded measures as initial data had to developed. We thus prove an interesting result of well-posedness for (1.1) for such initial data which extends to $p \in (p_c, 2)$ the existing one for the semilinear case $p = 2$ [6, 11] but holds true only if the singular diffusion equation $\partial_t v - \Delta_p v = 0$ in Q_∞ is well-posed in this setting. However, this issue seems to be still an open question for general non-negative and bounded measures but the answer is positive for Dirac masses which is exactly what is needed for the proof of Theorem 1.2.

Theorem 1.3. *Consider a non-negative bounded Borel measure $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$. If the singular diffusion equation*

$$\begin{aligned} \partial_t v - \Delta_p v &= 0 \quad \text{in} \quad Q_\infty\,, \\ v(0) &= u_0 \quad \text{in} \quad \mathbb{R}^N\,, \end{aligned}$$

has a unique solution $v \in C([0,\infty); \mathcal{M}_b^+(\mathbb{R}^N)) \cap C(Q_\infty)$, then there exists a unique non-negative function $u \in C(Q_\infty)$ which is a viscosity solution to (1.1) *in Q_∞ and satisfies*

$$\lim_{t\to 0} \int_{\mathbb{R}^N} \psi(x)\ u(t,x)\, dx = \int_{\mathbb{R}^N} \psi(x)\ du_0(x) \tag{1.10}$$

for any bounded and continuous function $\psi \in BC(\mathbb{R}^N)$. Moreover, $u(t)$ belongs to $L^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ for all $t > 0$ and satisfies

$$\|u(t)\|_1 \le M_0 := \int_{\mathbb{R}^N} du_0(x)\,, \tag{1.11}$$

as well as the following estimates

$$\|u(t)\|_1 + t^{N\eta} \|u(t)\|_\infty \le C_s(M_0)\,, \tag{1.12}$$

and

$$\|\nabla u(t)\|_\infty \le C_s(M_0) \left(1 + t^{(N+1)(q_*-q)\eta/(p-q)}\right) t^{-(N+1)\eta}\,, \tag{1.13}$$

where $C_s \in C([0,\infty))$ is a positive function depending only on N, p, and q.

The proof of this theorem is technical and quite involved, as usual when dealing with measures, since the lack of regularity does not allow to apply some of the standard techniques. In particular, Theorem 1.3 also implies the existence and uniqueness of a fundamental solution with any given mass $M > 0$ to (1.1), as it is explained at the end of Section 3.

Organisation of the paper. We collect in Section 2 many technical results and estimates needed in the sequel, in the form of separate lemmas. These include: a rigorous definition of viscosity solutions, decay estimates, estimates on the tail of the solution at sufficiently large times, and estimates of the solutions for small times, which are useful tools for identifying the initial trace. We agree that this section is a bit technical, but this allows us to state more clearly the main ideas and steps in the proofs of our main results. A reader who is not so interested in technical details could skip this part and admit the technical lemmas, or come back to it later.

Section 3 is devoted to the proof of Theorem 1.3. The proof is divided into two steps: we first construct a solution to (1.1) by classical approximation arguments. We next proceed to show the uniqueness of the solution which is actually the main contribution of this section. We then pass to the proof of Theorem 1.2, which occupies almost all Section 4 and is divided into several steps: we first construct a maximal and a minimal element in the class of the very singular solutions to (1.1). Then, we find that these two solutions are identical, by identifying both of them with the unique radially symmetric and self-similar very singular solution U, and we end up with the proof of the comparison principle for the associated elliptic equation. We end the paper with the proof of Theorem 1.1, to which Section 5 is devoted. It relies on the half-relaxed limits technique and is rather short, since most of the needed technical facts were already done in previous sections.

2 Well-posedness and decay estimates

In this section, we collect previous results on the well-posedness of (1.1) as well as some qualitative properties of the solutions. Let us first recall the notion of solutions we use throughout the paper.

2.1 Viscosity solution

As in our previous works [22, 23], a suitable notion of solution for equation (1.1) is that of *viscosity solution*, which is useful in dealing with the gradient term. Due to the singular character of (1.1) at points where ∇u vanishes, the standard definition of viscosity solution has to be adapted to deal with this case [25, 26, 30]. In fact, it requires to restrict the class of comparison functions [25, 30]. More precisely, let $\mathcal{F}$ be the set of functions $f \in C^2([0,\infty))$ satisfying

$$f(0) = f'(0) = f''(0) = 0, \ \ f''(r) > 0 \text{ for all } r > 0, \quad \lim_{r\to 0} |f'(r)|^{p-2} f''(r) = 0.$$

For example, $f(r) = r^\sigma$ with $\sigma > p/(p-1) > 2$ belongs to $\mathcal{F}$. We then introduce the class $\mathcal{A}$ of admissible comparison functions ψ defined as follows: a function $\psi \in C^2(Q_\infty)$ belongs to $\mathcal{A}$ if, for any $(t_0, x_0) \in Q_\infty$ where $\nabla\psi(t_0, x_0) = 0$, there exist a constant $\delta > 0$, a function $f \in \mathcal{F}$, and a modulus of continuity $\omega \in C([0,\infty))$, (that is, a non-negative function satisfying $\omega(r)/r \to 0$ as $r \to 0$), such that, for all $(t, x) \in Q_\infty$ with $|x - x_0| + |t - t_0| < \delta$, we have

$$|\psi(t, x) - \psi(t_0, x_0) - \partial_t\psi(t_0, x_0)(t - t_0)| \le f(|x - x_0|) + \omega(|t - t_0|).$$

With these notations, viscosity solutions to (1.1) are defined as follows [25, 26, 30]:

Definition 2.1. *An upper semicontinuous function $u : Q_\infty \to \mathbb{R}$ is a viscosity subsolution to (1.1) in Q_∞ if, whenever $\psi \in \mathcal{A}$ and $(t_0, x_0) \in Q_\infty$ are such that*

$$u(t_0, x_0) = \psi(t_0, x_0), \quad u(t, x) < \psi(t, x) \text{ for all } (t, x) \in Q_\infty \setminus \{(t_0, x_0)\},$$

then

$$\begin{cases} \partial_t\psi(t_0, x_0) \le \Delta_p\psi(t_0, x_0) - |\nabla\psi(t_0, x_0)|^q & \text{if } \nabla\psi(t_0, x_0) \ne 0, \\ \partial_t\psi(t_0, x_0) \le 0 & \text{if } \nabla\psi(t_0, x_0) = 0. \end{cases}$$

A lower semicontinuous function $u : Q_\infty \to \mathbb{R}$ is a viscosity supersolution to (1.1) in Q_∞ if, whenever $\psi \in \mathcal{A}$ and $(t_0, x_0) \in Q_\infty$ are such that

$$u(t_0, x_0) = \psi(t_0, x_0), \quad u(t, x) > \psi(t, x) \text{ for all } (t, x) \in Q_\infty \setminus \{(t_0, x_0)\},$$

then

$$\begin{cases} \partial_t\psi(t_0,x_0) \geq \Delta_p\psi(t_0,x_0) - |\nabla\psi(t_0,x_0)|^q & \text{if } \nabla\psi(t_0,x_0) \neq 0, \\ \partial_t\psi(t_0,x_0) \geq 0 & \text{if } \nabla\psi(t_0,x_0) = 0. \end{cases}$$

A continuous function $u : Q_\infty \to \mathbb{R}$ *is a viscosity solution to* (1.1) *in* Q_∞ *if it is a viscosity subsolution and supersolution.*

A remarkable feature of this modified definition is that basic results about viscosity solutions, such as comparison principle and stability property, are still valid, see [30, Theorem 3.9] (comparison principle) and [30, Theorem 6.1] (stability). The relationship between viscosity solutions and other notions of solutions is investigated in [26]. From now on, by a solution to (1.1) we mean a viscosity solution in the sense of Definition 2.1 above.

With this notion of solution to (1.1), we have the following well-posedness result [22, Theorem 6.2].

Proposition 2.2. *Assume that* u_0 *is a function satisfying the conditions* (1.7)*. Then there exists a unique non-negative function* $u \in C([0,\infty) \times \mathbb{R}^N)$ *which is a viscosity solution to* (1.1) *in* Q_∞ *and satisfies* $u(0) = u_0$*. In addition,* $u(t) \in L^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ *for each* $t > 0$ *and* u *is also a weak solution to* (1.1)*-*(1.2) *in the following sense:*

$$\int_{\mathbb{R}^N} (u(t,x) - u(s,x))\psi(x)\,dx + \int_s^t \int_{\mathbb{R}^N} \left(|\nabla u|^{p-2}\nabla u \cdot \nabla\psi + |\nabla u|^q\psi\right)\,dx\,d\tau = 0, \qquad (2.1)$$

for any $0 \leq s < t < \infty$ *and* $\psi \in C_0^\infty(\mathbb{R}^N)$*.*

As usual for homogeneous parabolic equations, the radial symmetry and monotonicity are preserved, as the following result states.

Lemma 2.3. *If* u_0 *satisfies* (1.7) *and is radially symmetric and non-increasing with respect to* $|x|$*, then the same properties hold true for* $u(t)$*, for any* $t > 0$*.*

Proof. The radial symmetry of $u(t)$ for positive times $t > 0$ follows readily from the rotational invariance of (1.1) and the well-posedness of (1.1). Next, we can write $u(t,x) = u(t,|x|) = u(t,r)$, and it satisfies

$$\partial_t u - (p-1)|\partial_r u|^{p-2}\partial_r^2 u - \frac{N-1}{r}|\partial_r u|^{p-2}\partial_r u + |\partial_r u|^q = 0.$$

At a formal level, it is clear that the zero function is a solution to the equation satisfied by $\partial_r u$ (which can be derived by differentiating the above equation for u), and the claimed monotonicity follows from the comparison principle since $\partial_r u_0 \leq 0$. Thanks to the uniqueness of solutions to (1.1), this argument can be made rigorous by standard approximations, as in [22]. □

A classical property of parabolic equations is that a modulus of continuity in space entails a modulus of continuity in time. In that direction, we have the following result which can be proved as [18, Lemma 5].

Lemma 2.4. *Consider an initial condition* u_0 *satisfying* (1.7) *and let* u *be the corresponding solution to* (1.1)*-*(1.2)*. Assume further that there are* $\tau \geq 0$ *and* $A > 0$ *such that* $\|\nabla u(t)\|_\infty \leq A$ *for all* $t \in [\tau,\infty)$*. Then there is* $C_2 > 0$ *depending only on* N*,* p*, and* q *such that*

$$|u(t,x) - u(s,x)| \leq C_2\,\left[(1+A)\ |t-s|^{1/2} + A^q\ |t-s|\right], \qquad t > s \geq \tau. \qquad (2.2)$$

2.2 Decay estimates

We next recall temporal decay estimates in $L^1(\mathbb{R}^N)$ and $W^{1,\infty}(\mathbb{R}^N)$ which are consequences of the analysis performed in [22] and depend on the behavior of the initial data as $|x| \to \infty$.

Proposition 2.5. *Assume that u_0 satisfies* (1.7) *and denote the corresponding solution to* (1.1)-(1.2) *by u. Then there is a constant $C>0$ depending only on N, p, and q such that*

$$|\nabla u(t,x)| \le C \left(\|u(s)\|_\infty^{1/\alpha p} + (t-s)^{-1/p} \right) (u(t,x))^{2/p} , \quad 0 \le s < t , \ \ x \in \mathbb{R}^N . \tag{2.3}$$

In addition, if M is such that $M \ge \|u_0\|_1$, then the estimates (1.12) *and* (1.13) *hold true with $C_s(M)$ instead of $C_s(M_0)$.*

Proof. The estimate (2.3) is a straightforward consequence of [22, Theorem 1.3 (i) & (ii)], while (1.12) follows by comparison with the solution v to the diffusion equation

$$\begin{aligned} \partial_t v - \Delta_p v &= 0 \quad \text{in} \quad Q_\infty , & (2.4)\\ v(0) &= u_0 \quad \text{in} \quad \mathbb{R}^N , & (2.5) \end{aligned}$$

see [17] for instance. Indeed, we obviously have $u \le v$ in Q_∞ by the comparison principle and, since $p > p_c$, we deduce from [17, Lemma III.6.1 & Theorem III.6.2] (with $r=1$ and $R=\infty$) that

$$\|v(t)\|_1 \le C \ \|u_0\|_1 \quad \text{and} \quad \|v(t)\|_\infty \le C \ \|u_0\|_1^{p\eta} \ t^{-N\eta} \tag{2.6}$$

for $t>0$. Finally, (1.13) readily follows from (2.3) (with $s=t/2$) and (1.12). □

For initial data decaying sufficiently fast as $|x| \to \infty$, faster temporal decay estimates were also supplied in [22, Theorem 1.2], which are only valid when p and q satisfy (1.3).

Proposition 2.6. *Assume that u_0 satisfies* (1.7) *as well as*

$$0 \le u_0(x) \le \kappa \ |x|^{-\alpha/\beta} , \quad x \in \mathbb{R}^N , \tag{2.7}$$

for some $\kappa>0$, and denote the corresponding solution to (1.1)-(1.2) *by u. Then there is a constant $K_\kappa > 0$ depending only on N, p, q, and κ such that*

$$t^{\alpha - N\beta} \|u(t)\|_1 + t^\alpha \|u(t)\|_\infty + t^{\alpha+\beta} \|\nabla u(t)\|_\infty \le K_\kappa, \quad t>0. \tag{2.8}$$

The precise dependence of K_κ on the parameters is not stated in [22, Theorem 1.2 (i)] but can be recovered by inspecting the proofs of [22, Theorem 1.2 (i) & Lemma 5.1].

2.3 Small time estimates

The previous decay estimates allow us to analyze precisely the behavior of solutions to (1.1) for small times, a fact which will be of utmost importance when considering non-smooth or even singular initial data.

Proposition 2.7. *Assume that u_0 satisfies* (1.7) *and denote the corresponding solution to* (1.1)-(1.2) *by u.*

(a) Let $\psi \in C_0^\infty(\mathbb{R}^N)$ and $T>0$. If M is such that $M \ge \|u_0\|_1$, there exists a constant $C(M,T)>0$ depending only on N, p, q, M, and T such that, for $t\in(0,T)$,

$$\begin{aligned}&\left|\int_{\mathbb{R}^N} (u(t,x)-u_0(x))\ \psi(x)\,dx\right| \\ \le& C(M,T)\left[\|\psi\|_\infty\ t^{(N+1)(q_*-q)\eta} + \|\nabla\psi\|_{p/(2-p)}\ t^{1/p}\right]\,.\end{aligned} \tag{2.9}$$

(b) Let $\psi \in C_0^\infty(\mathbb{R}^N)$ be a non-negative function such that $\psi(x)=0$ for $x\in B_r(0)$ for some $r>0$. If u_0 satisfies (2.7) for some $\kappa>0$, there exists a constant $C(\kappa,r)>0$ depending only on N, p, q, κ, and r such that, for $t>0$,

$$\int_{\mathbb{R}^N} u(t,x)\ \psi(x)\,dx \le \int_{\mathbb{R}^N} u_0(x)\ \psi(x)\,dx + C(\kappa,r)\|\nabla\psi\|_{p/(2-p)}\ t^{1/p}\,. \tag{2.10}$$

Proof. **Case (a).** Let $\psi\in C_0^\infty(\mathbb{R}^N)$, $T>0$, and $t\in(0,T)$. It follows from (2.1) that

$$\begin{aligned}&\left|\int_{\mathbb{R}^N} (u(t,x)-u_0(x))\ \psi(x)\,dx\right| \\ \le& \int_0^t\int_{\mathbb{R}^N} \left(|\nabla u(s,x)|^{p-1}\ |\nabla\psi(x)| + |\nabla u(s,x)|^q\ \psi(x)\right)\,dx\,ds.\end{aligned} \tag{2.11}$$

To estimate the gradient terms in the right-hand side of (2.11), we first notice that (2.3) and (1.12) give for $(s,x)\in Q_\infty$

$$\begin{aligned}|\nabla u(s,x)| &\le C\left[\left\|u\left(\frac{s}{2}\right)\right\|_\infty^{1/\alpha p} + s^{-1/p}\right](u(s,x))^{2/p}\,, \\ &\le C(M)\left[s^{-N\eta/\alpha p} + s^{-1/p}\right](u(s,x))^{2/p}\,. \end{aligned}\tag{2.12}$$

Now, we infer from (1.12) and (2.12) that

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u(s,x)|^q\ |\psi(x)|\,dx \le& C(M)\ \|\psi\|_\infty\left[s^{-qN\eta/\alpha p}+s^{-q/p}\right]\|u(s)\|_\infty^{(2q-p)/p}\ \|u(s)\|_1 \\ \le& C(M)\ \|\psi\|_\infty\left[s^{-N\eta/\alpha}+s^{-((N+1)q-N)\eta}\right].\end{aligned}$$

Observing that

$$\begin{aligned}&1-\frac{N\eta}{\alpha} = \frac{(N+1)(q_*-q)p\eta}{p-q}>0, \\ &1-((N+1)q-N)\eta = (N+1)(q_*-q)\eta>0\end{aligned}$$

by (1.3), we integrate the above inequality over $(0,t)$ and obtain

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^N} |\nabla u(s,x)|^q|\psi(x)|\,dx\,ds \\ \le& C(M)\ \|\psi\|_\infty\left[t^{(N+1)(q_*-q)p\eta/(p-q)} + t^{(N+1)(q_*-q)\eta}\right] \\ \le& C(M)\ \|\psi\|_\infty\left[1+t^{(N+1)(q_*-q)q\eta/(p-q)}\right]t^{(N+1)(q_*-q)\eta}.\end{aligned} \tag{2.13}$$

Similarly, by (2.12) and Hölder's inequality,

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u(s,x)|^{p-1}\ |\nabla\psi(x)|\,dx \\
&\le C(M)\left[s^{-(p-1)N\eta/\alpha p} + s^{-(p-1)/p}\right] \int_{\mathbb{R}^N} (u(s,x))^{2(p-1)/p}\ |\nabla\psi(x)|\,dx \\
&\le C(M)\left[s^{-(p-1)N\eta/\alpha p} + s^{-(p-1)/p}\right] \|u(s)\|_1^{2(p-1)/p}\ \|\nabla\psi\|_{p/(2-p)} \\
&\le C(M)\left[1 + s^{(p-1)(N+1)(q_*-q)\eta/(p-q)}\right] \|\nabla\psi\|_{p/(2-p)}\ s^{-(p-1)/p},
\end{aligned}$$

hence, after integrating over $(0,t)$,

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^N} |\nabla us,x)|^{p-1}\ |\nabla\psi(x)|\,dx\,ds \\
\le& C(M)\left[1 + t^{(p-1)(N+1)(q_*-q)\eta/(p-q)}\right] \|\nabla\psi\|_{p/(2-p)}\ t^{1/p}.
\end{aligned} \tag{2.14}$$

Combining (2.11), (2.13), and (2.14) gives (2.9).

Case (b). Let $t>0$ and a non-negative function $\psi \in C_0^\infty(\mathbb{R}^N)$. Since u_0 satisfies (2.7), it follows from (2.3) and (2.8) that, for $(s,x)\in Q_\infty$,

$$\begin{aligned}
|\nabla u(s,x)| &\le C\left[\left\|u\left(\frac{s}{2}\right)\right\|_\infty^{1/\alpha p} + s^{-1/p}\right] (u(s,x))^{2/p}\ , \\
&\le C(\kappa)\ s^{-1/p}\,(u(s,x))^{2/p}\ .
\end{aligned} \tag{2.15}$$

Owing to the non-negativity of ψ, it follows from (2.1) and (2.15) that

$$\begin{aligned}
\int_{\mathbb{R}^N} (u(t,x)-u_0(x))\ \psi(x)\,dx \le& \int_0^t \int_{\mathbb{R}^N} |\nabla u(s,x)|^{p-1}\ |\nabla\psi(x)|\,dx\,ds \\
\le& C(\kappa) \int_0^t \int_{\mathbb{R}^N} (u(s,x))^{2(p-1)/p}\ |\nabla\psi(x)|\ s^{-(p-1)/p}\,dx\,ds\,.
\end{aligned}$$

We now use again the decay property (2.7) of u_0 together with [22, Equation (5.5)] to conclude that $u(s,x)\le C(\kappa)\ |x|^{-\alpha/\beta}$ for $(s,x)\in Q_\infty$. Since ψ vanishes in $B_r(0)$ then so does $\nabla\psi$ and, by Hölder's inequality,

$$\begin{aligned}
\int_{\mathbb{R}^N} (u(t,x)-u_0(x))\ \psi(x)\,dx \le& C(\kappa) \int_0^t \int_{\{|x|>r\}} |x|^{-2(p-1)\alpha/p\beta}\ |\nabla\psi(x)|\ s^{-(p-1)/p}\,dx\,ds \\
\le& C(\kappa)\ t^{1/p} \left(\int_{\{|x|>r\}} |x|^{-\alpha/\beta}\,dx\right)^{2(p-1)/p} \|\nabla\psi\|_{p/(2-p)}\,,
\end{aligned}$$

from which (2.10) follows since $\alpha/\beta > N$ by (1.5). □

2.4 Tail behavior

We end this section with a control on the tail of solutions to (1.1)-(1.2). We first establish a pointwise estimate by showing the existence of a universal upper bound (also refered to as a *friendly giant* in literature), an idea also used in previous works, see [5, 7, 27, 34] for instance. We define

$$\Gamma_{p,q}(r) := \gamma\ r^{-\alpha/\beta}, \quad r>0, \tag{2.16}$$

where

$$\gamma := \frac{q-p+1}{p-q}\ \left(\frac{p-1}{q-p+1}\right)^{1/(q-p+1)}\,, \tag{2.17}$$

and first state some useful properties of $\Gamma_{p,q}$.

Lemma 2.8. *For all $r>0$, $\Gamma_{p,q}$ belongs to $L^1(\mathbb{R}^N\setminus B_r(0))$ and $(t,x)\longmapsto \Gamma_{p,q}(|x|-r)$ is a supersolution to* (1.1) *in $(0,\infty)\times(\mathbb{R}^N\setminus B_r(0))$.*

Proof. The stated integrability of $\Gamma_{p,q}$ follows from the property $\alpha/\beta>N$, see (1.5), while a direct computation and the monotonicity of $\Gamma_{p,q}$ give the second assertion. □

Lemma 2.9. *Consider an initial condition u_0 satisfying* (1.7) *and let u be the corresponding solution to* (1.1)-(1.2)*. Define*

$$R(u_0) := \inf\Big\{R>0:\ u_0(x)|x|^{\alpha/\beta}\le\gamma\ a.e.\ in\ \{|x|\ge R\}\Big\}\in[0,\infty]. \tag{2.18}$$

If $R(u_0)<\infty$, then

$$0\le u(t,x)\le\Gamma_{p,q}(|x|-R(u_0)) \tag{2.19}$$

for any $t>0$ and $x\in\mathbb{R}^N$ with $|x|>R(u_0)$.

Proof. Clearly,

$$u_0(x)\le\gamma|x|^{-\alpha/\beta}=\Gamma_{p,q}(|x|-R(u_0))\,,\qquad x\in\mathbb{R}^N\setminus B_{R(u_0)}(0)\,.$$

In addition, for all $x\in\mathbb{R}^N$ such that $|x|=R(u_0)$ and $t>0$, we have $\Gamma_{p,q}(|x|-R(u_0))=\infty>u(t,x)$. Thus, $u(t,x)\le\Gamma_{p,q}(|x|-R(u_0))$ on the parabolic boundary of $(0,\infty)\times(\mathbb{R}^N\setminus B_{R(u_0)}(0))$, and the comparison principle guarantees that $u(t,x)\le\Gamma_{p,q}(|x|-R(u_0))$ in $[0,\infty)\times\mathbb{R}^N\setminus B_{R(u_0)}(0)$. □

We next prove an integral estimate on the tail behaviour of solutions to (1.1)-(1.2).

Lemma 2.10. *Let u_0 be an initial condition satisfying* (1.7) *and denote the corresponding solution to* (1.1)-(1.2) *by u. There is $C_0>0$ depending only on N, p, and q such that, for $R>0$ and $t\ge0$, there holds*

$$\int_{\{|x|\ge R\}} u(t,x)\,dx\le C_0\ R^{(\beta N-\alpha)/\beta}\ \left(\sup_{|x|\ge R/2}\Big\{u_0(x)\ |x|^{\alpha/\beta}\Big\}+t\ R^{-1/\beta}\right)\,. \tag{2.20}$$

Proof. We fix $\zeta\in C^\infty(\mathbb{R}^N)$ such that $0\le\zeta\le1$ and

$$\zeta(x)=0\quad\text{if}\quad |x|\le\frac{1}{2}\qquad\text{and}\qquad \zeta(x)=1\quad\text{if}\quad |x|\ge1\,. \tag{2.21}$$

For $R>0$ and $x\in\mathbb{R}^N$, we define $\zeta_R(x):=\zeta(x/R)$. It follows from the weak formulation of (1.1) and Young's inequality that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^N}\zeta_R(x)^{q/(q-p+1)}\ u(t,x)\,dx+\int_{\mathbb{R}^N}\zeta_R(x)^{q/(q-p+1)}\ |\nabla u(t,x)|^q\,dx\\
\le&\frac{q}{q-p+1}\ \int_{\mathbb{R}^N}\zeta_R(x)^{(p-1)/(q-p+1)}\ |\nabla u(t,x)|^{p-1}\ |\nabla\zeta_R(x)|\,dx\\
\le&\frac{p-1}{q-p+1}\ \int_{\mathbb{R}^N}\zeta_R(x)^{q/(q-p+1)}\ |\nabla u(t,x)|^q\,dx+\int_{\mathbb{R}^N}|\nabla\zeta_R(x)|^{q/(q-p+1)}\,dx\,,
\end{aligned}$$

whence

$$\frac{d}{dt}\int_{\mathbb{R}^N} \zeta_R(x)^{q/(q-p+1)}\ u(t,x)\,dx \le C(\zeta)\ R^{(\beta N-\alpha-1)/\beta}\,. \tag{2.22}$$

Owing to the properties (2.21) of ζ, we find, after integrating with respect to time,

$$\begin{aligned}\int_{\{|x|\ge R\}} u(t,x)\,dx &\le \int_{\mathbb{R}^N} \zeta_R(x)^{q/(q-p+1)}\ u(t,x)\,dx\\ &\le \int_{\mathbb{R}^N} \zeta_R(x)^{q/(q-p+1)}\ u_0(x)\,dx + C(\zeta)\ t\ R^{(\beta N-\alpha-1)/\beta}\\ &\le \sup_{|x|\ge R/2}\left\{u_0(x)\ |x|^{\alpha/\beta}\right\}\ \int_{\{|x|\ge R/2\}} |x|^{-\alpha/\beta}\,dx + C(\zeta)\ t\ R^{(\beta N-\alpha-1)/\beta}\,,\end{aligned}$$

from which (2.20) follows. $\square$

As a consequence of these integral tail estimates, we obtain some precise pointwise estimates for sufficiently rapidly decaying initial data.

Lemma 2.11. *If u_0 satisfies* (1.7) *and* (2.7) *for some $\kappa > 0$ and u denotes the corresponding solution to the Cauchy problem* (1.1)-(1.2), *then there exists $C > 0$ depending on N, p, and q such that*

$$|x|^{\alpha/\beta}u(t,x) \le C\left(\sup_{|y|\ge|x|/4}\{u_0(y)|y|^{\alpha/\beta}\} + t|x|^{-1/\beta}\right) \tag{2.23}$$

for any $x\in\mathbb{R}^N\setminus\{0\}$ and $t>0$.

Proof. **Step 1.** Let first u_0 be radially symmetric and non-increasing with respect to $|x|$. Then, by Lemma 2.3, $u(t)$ has the same properties for any $t>0$, and for $x\in\mathbb{R}^N$, $x\ne 0$ we deduce from Lemma 2.10 that

$$\begin{aligned}Cu(t,x)|x|^N &\le \int_{\{|x|/2\le|y|\le|x|\}} u(t,y)\,dy\\ &\le C_0\left(\frac{|x|}{2}\right)^{(N\beta-\alpha)/\beta}\left(\sup_{|y|\ge|x|/4}\{u_0(y)|y|^{\alpha/\beta}\} + t\left(\frac{2}{|x|}\right)^{1/\beta}\right)\\ &\le 2^{(1+\alpha)/\beta}C_0|x|^{(N\beta-\alpha)/\beta}\left(\sup_{|y|\ge|x|/4}\{u_0(y)|y|^{\alpha/\beta}\} + t|x|^{-1/\beta}\right).\end{aligned}$$

which gives (2.23) for this specific class of initial data.

Step 2. Fix $x_0\in\mathbb{R}^N\setminus\{0\}$. We define

$$\kappa_0 := \sup_{|y|\ge|x_0|/4}\{u_0(y)|y|^{\alpha/\beta}\} \le \kappa$$

and take $R_0\in(0,|x_0|/4)$ such that $\kappa_0 R_0^{-\alpha/\beta} \ge \|u_0\|_\infty$. We define

$$\tilde{u}_0(x) := \begin{cases} 2\kappa_0|x|^{-\alpha/\beta}, & |x|\ge R_0,\\ 2\kappa_0 R_0^{-\alpha/\beta}, & |x|\le R_0.\end{cases} \tag{2.24}$$

Then $\tilde{u}_0$ is a radially symmetric and non-increasing function of $|x|$ and it satisfies (1.7) since $\alpha/\beta > N$ as well as (2.7) with constant $2\kappa_0$. Moreover, $u_0 \le \tilde{u}_0$ in $\mathbb{R}^N$, hence the comparison principle guarantees that $u \le \tilde{u}$ in Q_∞, where $\tilde{u}$ is the solution to (1.1) with initial condition $\tilde{u}_0$. Applying Step 1 above to $\tilde{u}$ gives

$$\begin{aligned}|x_0|^{\alpha/\beta} u(t,x_0) \le & |x_0|^{\alpha/\beta} \tilde{u}(t,x_0) \le 2^{(1+\alpha)/\beta} C_0 \left(\sup_{|y|\ge |x_0|/4} \{\tilde{u}_0(y)|y|^{\alpha/\beta}\} + t|x_0|^{-1/\beta} \right) \\ \le & 2^{(1+\alpha)/\beta} C_0 \left(2\kappa_0 + t|x_0|^{-1/\beta} \right) ,\end{aligned}$$

and thus (2.23). □

3 Well-posedness with non-negative bounded measures as initial data

In this section, we prove Theorem 1.3, together with some preparatory results. We begin with the proof of the existence statement which will be done, as usual, through an approximation process.

Proof of Theorem 1.3. Existence. Let $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$ and $(u_0^k)_{k\ge 1}$ be a sequence of functions in $C_0^\infty(\mathbb{R}^N)$ such that

$$\|u_0^k\|_1 = M_0 := \int_{\mathbb{R}^N} du_0 \,, \tag{3.1}$$

and

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} u_0^k(x)\psi(x)\, dx = \int_{\mathbb{R}^N} \psi(x)\, du_0(x) \quad \text{for any } \psi \in BC(\mathbb{R}^N). \tag{3.2}$$

Given $k \ge 1$, we denote the unique solution of (1.1) with initial condition u_0^k by u^k. Owing to (3.1), it follows from Proposition 2.5 that $(u^k)_k$ is bounded in $L^\infty(\tau,\infty; W^{1,\infty}(\mathbb{R}^N))$ for each $\tau > 0$. Combining this property with Lemma 2.4 implies the time equicontinuity of the sequence $(u^k)_k$ in $(\tau,\infty)\times\mathbb{R}^N$ for all $\tau > 0$. We then deduce from the Arzelà-Ascoli theorem that $(u^k)_k$ is relatively compact in $C([\tau,T]\times K)$ for all compact subsets K of $\mathbb{R}^N$ and $0 < \tau < T$. There are thus a subsequence (u^k) (not relabeled) and a continuous function $u \in C(Q_\infty)$ such that

$$u^k \longrightarrow u \quad \text{in} \quad C([\tau,T]\times K) \quad \text{as} \quad k\to\infty \tag{3.3}$$

for all compact subsets K of $\mathbb{R}^N$ and $0 < \tau < T$. Owing to the stability of viscosity solutions to (1.1) [30, Theorem 6.1], this convergence guarantees that u is a viscosity solution to (1.1) in Q_∞. In addition, since u^k satisfies (1.12) and (1.13) with the constant $C_s(M_0)$, so does u. Consequently, $u(t)$ belongs to $L^1(\mathbb{R}^N)$ and $W^{1,\infty}(\mathbb{R}^N)$ for all $t > 0$.

In order to complete the proof of the existence part, it remains to identify the initial condition taken by u. Consider $t \in (0,1)$, $\psi \in C_0^\infty(\mathbb{R}^N)$, and $k \ge 1$. Owing to (3.1), we are in a position to apply Proposition 2.7 (a) and conclude that

$$\begin{aligned}&\left| \int_{\mathbb{R}^N} u^k(t,x)\psi(x)\, dx - \int_{\mathbb{R}^N} u_0^k(x)\psi(x)\, dx \right| \\ &\qquad \le C(M_0,1) \left(t^{1/p} \, \|\nabla\psi\|_{p/(2-p)} + t^{(N+1)(q_*-q)\eta} \, \|\psi\|_\infty \right) .\end{aligned} \tag{3.4}$$

Owing to (3.2) and (3.3), we may let $k \to \infty$ in (3.4) to get

$$\left| \int_{\mathbb{R}^N} u(t,x)\psi(x)\, dx \; - \int_{\mathbb{R}^N} \psi(x)\, du_0(x) \right| \\ \leq C \; \left(t^{1/p} \|\nabla\psi\|_{p/(2-p)} + t^{(N+1)(q_*-q)\eta} \|\psi\|_\infty \right),$$

from which we readily deduce that

$$\lim_{t\to 0} \int_{\mathbb{R}^N} u(t,x)\psi(x)\, dx = \int_{\mathbb{R}^N} \psi(x)\, du_0(x) \tag{3.5}$$

for any $\psi \in C_0^\infty(\mathbb{R}^N)$. In fact, by a classical density argument, (3.5) is valid for any continuous function $\psi \in C_0(\mathbb{R}^N)$ which vanishes as $|x| \to \infty$. Let us now show that (3.5) is satisfied for any function $\psi \in BC(\mathbb{R}^N)$. To this end, let $\zeta \in C^\infty(\mathbb{R}^N)$ be such that $0 \leq \zeta \leq 1$ and

$$\zeta(x) = 0 \quad \text{if} \quad |x| \leq \frac{1}{2} \quad \text{and} \quad \zeta(x) = 1 \quad \text{if} \quad |x| \geq 1\,,$$

and $\psi \in BC(\mathbb{R}^N)$. Then, for $R>0$, $\left(1 - \zeta_R^{q/(q-p+1)}\right)\psi$ belongs to $C_0(\mathbb{R}^N)$ and

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N} u(t,x)\psi(x)\, dx - \int_{\mathbb{R}^N} \psi(x)\, du_0(x) \right| \\
\leq &\left| \int_{\mathbb{R}^N} u(t,x) \left(1 - \zeta_R(x)^{q/(q-p+1)}\right) \psi(x)\, dx - \int_{\mathbb{R}^N} \left(1 - \zeta_R(x)^{q/(q-p+1)}\right) \psi(x)\, du_0(x) \right| \\
&+ \int_{\mathbb{R}^N} u(t,x)\zeta_R(x)^{q/(q-p+1)}\psi(x)\, dx + \int_{\mathbb{R}^N} \zeta_R(x)^{q/(q-p+1)}\psi(x)\, du_0(x) \\
\leq &\left| \int_{\mathbb{R}^N} u(t,x) \left(1 - \zeta_R(x)^{q/(q-p+1)}\right) \psi(x)\, dx - \int_{\mathbb{R}^N} \left(1 - \zeta_R(x)^{q/(q-p+1)}\right) \psi(x)\, du_0(x) \right| \\
&+ \|\psi\|_\infty \left(\int_{\mathbb{R}^N} u(t,x)\zeta_R(x)^{q/(q-p+1)}\, dx + \int_{\mathbb{R}^N} \zeta_R(x)^{q/(q-p+1)}\, du_0(x) \right).
\end{aligned} \tag{3.6}$$

We now recall that it follows from (2.22) that

$$\int_{\mathbb{R}^N} u^k(t,x)\zeta_R(x)^{q/(q-p+1)}\, dx \leq \int_{\mathbb{R}^N} u_0^k(x)\zeta_R(x)^{q/(q-p+1)}\, dx + C(\zeta) t R^{(\beta N-\alpha-1)/\beta}$$

for $t \in (0,1)$ and $k \geq 1$. We then infer from (3.2), (3.3), and Fatou's lemma that

$$\int_{\mathbb{R}^N} u(t,x)\zeta_R(x)^{q/(q-p+1)}\, dx \leq \int_{\mathbb{R}^N} \zeta_R(x)^{q/(q-p+1)}\, du_0(x) + C(\zeta) t R^{(\beta N-\alpha-1)/\beta} \tag{3.7}$$

for $t \in (0,1)$. We then infer from (3.5), (3.6), and (3.7) that

$$\limsup_{t\to 0} \left| \int_{\mathbb{R}^N} u(t,x)\psi(x)\, dx - \int_{\mathbb{R}^N} \psi(x)\, du_0(x) \right| \leq 2\|\psi\|_\infty \int_{\mathbb{R}^N} \zeta_R(x)^{q/(q-p+1)}\, du_0(x). \tag{3.8}$$

Since u_0 is a bounded measure, we then let $R \to \infty$ in (3.8) and use the properties of ζ to conclude that the left-hand side of (3.8) vanishes. This ends the proof of the existence result. $\square$

We next turn to the proof of the uniqueness part of Theorem 1.3 for which the following two preliminary results are needed. We will first need the following inequality for vectors in $\mathbb{R}^N$.

Lemma 3.1. *If $q \geq p/2$, then there exists $\vartheta = \vartheta(p,q) \in (0,1]$ such that*

$$(a-b)\cdot\left(|a|^{p-2}a - |b|^{p-2}b\right) \geq \vartheta \frac{\left||a|^{q-1}a - |b|^{q-1}b\right|^2}{|a|^{2q-p} + |b|^{2q-p}} \geq \vartheta \frac{\left(|a|^q - |b|^q\right)^2}{|a|^{2q-p} + |b|^{2q-p}}, \tag{3.9}$$

for all $(a,b) \in \mathbb{R}^N \times \mathbb{R}^N$.

When $q = 1$ and $p \in (1,2]$, this lemma is proved in [13, Lemma A.2].

Proof. Consider $(a,b) \in \mathbb{R}^N \times \mathbb{R}^N$, $\vartheta \in (0,1]$, and define

$$\begin{aligned}
\Lambda(a,b) &:= (a-b)\cdot\left(|a|^{p-2}a - |b|^{p-2}b\right)\left(|a|^{2q-p} + |b|^{2q-p}\right) - \vartheta\left||a|^{q-1}a - |b|^{q-1}b\right|^2 \\
&= \left[|a|^p + |b|^p - \left(|a|^{p-2} + |b|^{p-2}\right)(a\cdot b)\right]\left(|a|^{2q-p} + |b|^{2q-p}\right) \\
&- \vartheta|a|^{2q} - \vartheta|b|^{2q} + 2\vartheta|a|^{q-1}|b|^{q-1}(a\cdot b) \\
&= \left(|a|^p + |b|^p\right)\left(|a|^{2q-p} + |b|^{2q-p}\right) - \vartheta\left(|a|^{2q} + |b|^{2q}\right) \\
&- \left[|a|^{2q-2} + |b|^{2q-2} + |a|^{p-2}|b|^{2q-p} + |a|^{2q-p}|b|^{p-2} - 2\vartheta|a|^{q-1}|b|^{q-1}\right](a\cdot b).
\end{aligned}$$

Since $\vartheta \in (0,1]$, we have

$$\begin{aligned}
&|a|^{2q-2} + |b|^{2q-2} + |a|^{p-2}|b|^{2q-p} + |a|^{2q-p}|b|^{p-2} - 2\vartheta|a|^{q-1}|b|^{q-1} \\
&\qquad \geq |a|^{2q-2} + |b|^{2q-2} - 2|a|^{q-1}|b|^{q-1} = \left(|a|^{q-1} - |b|^{q-1}\right)^2 \geq 0.
\end{aligned}$$

As $a\cdot b \leq |a||b|$, it follows from the previous inequalities that

$$\begin{aligned}
\Lambda(a,b) &\geq \left(|a|^p + |b|^p\right)\left(|a|^{2q-p} + |b|^{2q-p}\right) - \vartheta\left(|a|^{2q} + |b|^{2q}\right) \\
&- \left[|a|^{2q-2} + |b|^{2q-2} + |a|^{p-2}|b|^{2q-p} + |a|^{2q-p}|b|^{p-2} - 2\vartheta|a|^{q-1}|b|^{q-1}\right]|a||b| \\
&\geq \left(|a|^p + |b|^p - |a|^{p-1}|b| - |a||b|^{p-1}\right)\left(|a|^{2q-p} + |b|^{2q-p}\right) - \vartheta\left(|a|^q - |b|^q\right)^2 \\
&\geq \left(|a| - |b|\right)\left(|a|^{p-1} - |b|^{p-1}\right)\left(|a|^{2q-p} + |b|^{2q-p}\right) - \vartheta\left(|a|^q - |b|^q\right)^2.
\end{aligned}$$

Since $q \geq p/2$, it follows from [21, Lemma 1] that there is $C_1 \geq 1$ depending only on p and q such that

$$\frac{\left(|a|^q - |b|^q\right)^2}{\left(|a|^{p-1} - |b|^{p-1}\right)\left(|a| - |b|\right)} \leq C_1 \max\left\{|a|,|b|\right\}^{2q-p} \leq C_1\left(|a|^{2q-p} + |b|^{2q-p}\right).$$

Consequently, choosing $\vartheta = 1/C_1$, we end up with $\Lambda(a,b) \geq 0$, which implies the first inequality in (3.9). The second inequality then follows easily from the triangular inequality. $\square$

We next estimate the small time behavior of solutions to (1.1).

Lemma 3.2. *Consider $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$ and let u be a non-negative solution to (1.1) with initial condition u_0. If there exists a unique non-negative solution $v \in C([0,\infty);\mathcal{M}_b^+(\mathbb{R}^N))\cap$*

$C(Q_\infty)$ to the diffusion equation (2.4)-(2.5) *in Q_∞ with initial condition u_0, then, for $t>0$ and $r\in[1,\infty]$,*

$$\|u(t)\|_1 \le M_0 := \int_{\mathbb{R}^N} du_0(x)\,, \tag{3.10}$$

and

$$\|u(t)-v(t)\|_r \le C(M_0)\left(1+t^{(N+1)(q_*-q)q\eta/r(p-q)}\right)\ t^{[(N+1)(q_*-q)-N(r-1)]\eta/r}\,. \tag{3.11}$$

Proof. For $\tau>0$, let v^τ be the solution to the diffusion equation (2.4) in $(\tau,\infty)\times\mathbb{R}^N$ with initial condition $v^\tau(\tau)=u(\tau)$.

We first prove (3.10). By the comparison principle, $u\le v^\tau$ in $(\tau,\infty)\times\mathbb{R}^N$ while the L^1-accretivity of the p-Laplacian guarantees that $\|v^\tau(t)\|_1\le\|v^\tau(\tau)\|_1$ for $t>\tau$. Consequently, for $t>\tau$,

$$\|u(t)\|_1 \le \|v^\tau(t)\|_1 \le \|v^\tau(\tau)\|_1 = \int_{\mathbb{R}^N} u(\tau,x)\,dx \underset{\tau\to 0}{\longrightarrow} M_0\,,$$

and thus (3.10).

Next, since $u(\tau)\in L^1(\mathbb{R}^N)$ and $p>p_c$, it follows from the L^1-accretivity of the p-Laplacian that, for $t>\tau$,

$$\|u(t)-v^\tau(t)\|_1 \le \int_\tau^t \int_{\mathbb{R}^N} |\nabla u(s,x)|^q\,dx\,ds.$$

Thanks to (3.10), we may use (1.12) and (2.3) to obtain

$$\begin{aligned}
\|u(t)-v^\tau(t)\|_1 \le & C\ \int_\tau^t \int_{\mathbb{R}^N} \left[\left\|u\left(\frac{s+\tau}{2}\right)\right\|_\infty^{1/\alpha p} + (s-\tau)^{-1/p}\right]^q\ (u(s,x))^{2q/p}\ ds \\
\le & C(M_0)\ \int_\tau^t \left[(s-\tau)^{-qN\eta/\alpha p} + (s-\tau)^{-q/p}\right]\ \|u(s)\|_\infty^{(2q-p)/p}\ \|u(s)\|_1\,ds \\
\le & C(M_0)\ \int_\tau^t \left[(s-\tau)^{-qN\eta/\alpha p} + (s-\tau)^{-q/p}\right]\ s^{-(2q-p)N\eta/p}\,ds \\
\le & C(M_0)\ \int_\tau^t \left[(s-\tau)^{-N\eta/\alpha} + (s-\tau)^{-(q(N+1)-N)\eta}\right]\ ds \\
\le & C(M_0)\left[t^{(N+1)(q_*-q)p\eta/(p-q)} + t^{(N+1)(q_*-q)\eta}\right]\,,
\end{aligned}$$

hence

$$\|u(t)-v^\tau(t)\|_1 \le C(M_0)\left[1+t^{(N+1)(q_*-q)q\eta/(p-q)}\right]t^{(N+1)(q_*-q)\eta}\,, \qquad t>\tau\,.$$

Now, since $v^\tau(t)$ converges towards $v(t)$ in $L^1(\mathbb{R}^N)$ for all $t>0$, we conclude that

$$\|u(t)-v(t)\|_1 \le C(M_0)\left[1+t^{(N+1)(q_*-q)q\eta/(p-q)}\right]t^{(N+1)(q_*-q)\eta}\,, \qquad t>0\,. \tag{3.12}$$

Also, by (1.12), (2.6), and (3.10),

$$\|u(t)-v(t)\|_\infty \le \|u(t)\|_\infty + \|v(t)\|_\infty \le C(M_0)\ t^{-N\eta}\,, \quad t>0. \tag{3.13}$$

We then infer from (3.12), (3.13), and Hölder's inequality that (3.11) is true. $\square$

Proof of Theorem 1.3. Uniqueness. Let u_1 and u_2 be two non-negative solutions to (1.1) with initial condition $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$ and define

$$M_0 := \int_{\mathbb{R}^N} du_0(x)\,,$$

and $w := u_1 - u_2$. Then, w solves

$$\partial_t w - (\Delta_p u_1 - \Delta_p u_2) + |\nabla u_1|^q - |\nabla u_2|^q = 0 \quad \text{in } Q_\infty. \tag{3.14}$$

Consider $r > 0$ to be specified later and $T > 0$. For $t \in (0, T)$, we calculate

$$\begin{aligned}\frac{1}{r+1}\,\frac{d}{dt}\|w\|_{r+1}^{r+1} = &- \int_{\mathbb{R}^N} r|w|^{r-1}\nabla w \cdot \left(|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2\right)\,dx\\ &- \int_{\mathbb{R}^N} |w|^{r-1} w\left(|\nabla u_1|^q - |\nabla u_2|^q\right)\,dx.\end{aligned}$$

Lemma 3.1 then gives, with the help of Young's inequality,

$$\begin{aligned}&\frac{1}{r+1}\,\frac{d}{dt}\|w\|_{r+1}^{r+1}\\ \le& - r\vartheta \int_{\mathbb{R}^N} |w|^{r-1} \frac{(|\nabla u_1|^q - |\nabla u_2|^q)^2}{1 + |\nabla u_1|^{2q-p} + |\nabla u_2|^{2q-p}}\,dx\\ &+ \int_{\mathbb{R}^N} |w|^{(r+1)/2} \frac{|w|^{(r-1)/2}(|\nabla u_1|^q - |\nabla u_2|^q)}{\sqrt{1 + |\nabla u_1|^{2q-p} + |\nabla u_2|^{2q-p}}}\sqrt{1 + |\nabla u_1|^{2q-p} + |\nabla u_2|^{2q-p}}\,dx\\ \le& C \int_{\mathbb{R}^N} |w|^{r+1}\left(1 + |\nabla u_1|^{2q-p} + |\nabla u_2|^{2q-p}\right)\,dx\\ \le& C\left(1 + \|\nabla u_1\|_\infty^{2q-p} + \|\nabla u_2\|_\infty^{2q-p}\right)\|w\|_{r+1}^{r+1}.\end{aligned}$$

Owing to (3.10), we are in a position to use the gradient estimate (1.13) and we further obtain

$$\frac{1}{r+1}\,\frac{d}{dt}\|w(t)\|_{r+1}^{r+1} \le C(M_0, T)\left(1 + t^{-(N+1)\eta(2q-p)}\right)\|w(t)\|_{r+1}^{r+1}.$$

Observing that

$$1 - (N+1)\eta(2q-p) = 2(N+1)(q_* - q)\eta > 0,$$

we may integrate the above differential inequality over (s, t), $0 < s < t < T$, to obtain

$$\|w(t)\|_{r+1}^{r+1} \le \|w(s)\|_{r+1}^{r+1}\,\exp\left\{(r+1)C(M_0, T)\left(t^{2(N+1)(q_*-q)\eta} + t\right)\right\}. \tag{3.15}$$

We now choose $r \in (0, (N+1)(q_* - q)/N)$ and realize that (3.11) guarantees that (keeping the notation of Lemma 3.2)

$$\begin{aligned}\|w(s)\|_{r+1}^{r+1} \le& \|u_1(s) - v(s)\|_{r+1}^{r+1} + \|v(s) - u_2(s)\|_{r+1}^{r+1}\\ \le& C(M_0, T)\; s^{((N+1)(q_*-q)-Nr)\eta} \underset{s\to 0}{\longrightarrow} 0.\end{aligned}$$

Consequently, letting $s \to 0$ in (3.15) leads us to $\|w(t)\|_{r+1}^{r+1} \le 0$ for all $t \in (0, T)$, hence $u_1 \equiv u_2$ in $(0, T)$. As T was arbitrary, the proof is complete. $\square$

Since initial data of the form $M\delta_0$, where δ_0 denotes the Dirac mass at $x = 0$, play an essential role in the sequel, we rephrase Theorem 1.3 in this particular setting.

Corollary 3.3. *For any $M > 0$, there exists a unique solution u_M to* (1.1) *with initial condition $M\delta_0$. In the sequel, u_M will be refered to as* the fundamental solution to (1.1) of mass M. *Moreover, it satisfies the estimates* (1.12) *and* (1.13) *with $C_s(M)$ instead of $C_s(M_0)$.*

Proof. The existence and uniqueness of a solution for the p-Laplacian equation (2.4) with initial condition $u_0 = M\delta_0$ are proved in [15, Theorem 4.1]. Thus, applying Theorem 1.3 with $u_0 = M\delta_0$, we get the claimed result. □

4 Very singular solutions

As specified in the Introduction, we will study in detail the very singular solutions of (1.1). More precisely, in this section we show that there exists in fact a unique very singular solution to (1.1). This is done by constructing a minimal and a maximal very singular solution and identifying them afterwards. We begin with the precise definition.

Definition 4.1. *A very singular solution to* (1.1) *is a viscosity solution u to* (1.1) *in Q_∞ in the sense of Definition 2.1 satisfying*

$$u(t) \in L^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N) \tag{4.1}$$

for all $t > 0$ as well as

$$\lim_{s\to 0} \int_{\{|x|\le r\}} u(s,x)\,dx = \infty, \quad r \in (0,\infty), \tag{4.2}$$

and

$$\lim_{s\to 0} \int_{\{|x|\ge r\}} u(s,x)\,dx = 0, \quad r \in (0,\infty). \tag{4.3}$$

A very singular subsolution (resp. supersolution) to (1.1) *is a viscosity subsolution (resp. supersolution) to* (1.1) *in Q_∞ in the sense of Definition 2.1, which satisfies* (4.1), (4.2) *and* (4.3).

We already know that the class of very singular solutions to (1.1) for $p \in (p_c, 2)$ and $q \in (p/2, q_*)$ is non-empty as a consequence of the following result [23].

Theorem 4.2. *There exists a unique radially symmetric, self-similar very singular solution U to* (1.1), *having the form*

$$U(t,x) = t^{-\alpha} f_U(|x|t^{-\beta}), \quad (t,x) \in Q_\infty. \tag{4.4}$$

The profile f_U is a solution to the differential equation

$$(|f_U'|^{p-2}f_U')'(r) + \frac{N-1}{r}(|f_U'|^{p-2}f_U')(r) + \alpha f_U(r) + \beta r f_U'(r) - |f_U'(r)|^q = 0\,, \quad r > 0, \tag{4.5}$$

satisfying $f_U'(0) = 0$ and there is an explicit positive constant ω^* such that

$$\lim_{r\to\infty} r^{p/(2-p)}\, f_U(r) = \omega^*.$$

This important result is very useful in the sequel in order to identify very singular solutions when we are able to show that they are radially symmetric and in self-similar form.

4.1 Some properties of very singular subsolutions and solutions

From Definition 4.1, one expects the initial trace of a very singular solution to (1.1) to vanish outside the origin. This is made rigorous in the next result.

Proposition 4.3. *Let u be a very singular subsolution to* (1.1) *and K be a compact subset of $\mathbb{R}^N \setminus \{0\}$. Then*

$$\lim_{t\to 0} \sup_{x\in K}\{u(t,x)\} = 0.$$

Proof. Fix $\tau > 0$ and let v_τ be the solution to the diffusion equation (2.4) in $(\tau,\infty)\times\mathbb{R}^N$ with initial condition $v_\tau(\tau) = u(\tau)$. According to [17, Theorem III.6.2], v_τ satisfies the following pointwise estimate: there exists a constant $C > 0$ depending only on N and p such that, for any $x_0 \in \mathbb{R}^N$, $R > 0$, and $t > \tau$,

$$\sup_{x\in B_R(x_0)} \{v_\tau(t,x)\} \le C\ (t-\tau)^{-N\eta} \left(\int_{B_{2R}(x_0)} v_\tau(\tau,x)\,dx\right)^{p\eta} + C\left(\frac{t-\tau}{R^p}\right)^{1/(2-p)}. \tag{4.6}$$

Since u is a subsolution to the diffusion equation (2.4) in $(\tau,\infty)\times\mathbb{R}^N$ with $u(\tau) = v_\tau(\tau)$, the comparison principle gives $u \le v_\tau$ in $(\tau,\infty)\times\mathbb{R}^N$. Plugging these information in (4.6), we are led to

$$\sup_{x\in B_R(x_0)} \{u(t,x)\} \le C\ (t-\tau)^{-N\eta} \left(\int_{B_{2R}(x_0)} u(\tau,x)\,dx\right)^{p\eta} + C\left(\frac{t-\tau}{R^p}\right)^{1/(2-p)} \tag{4.7}$$

for any $t > \tau > 0$. Now, assume further that $x_0 \ne 0$ and $|x_0| > 2R$. Then $0 \notin B_{2R}(x_0)$ and we may let $\tau \to 0$ in (4.7) and use (4.3) to obtain

$$\sup_{x\in B_R(x_0)} \{u(t,x)\} \le C\left(\frac{t}{R^p}\right)^{1/(2-p)}, \qquad t > 0\,. \tag{4.8}$$

Therefore, if $x_0 \ne 0$ and $|x_0| > 2R$,

$$\lim_{t\to 0} \sup_{x\in B_R(x_0)} \{u(t,x)\} = 0,$$

and this property entails Proposition 4.3 by a covering argument. □

In particular, Proposition 4.3 implies that $u(0,x) = 0$, for any very singular subsolution u and any $x \ne 0$. This is useful to prove some comparison results.

Proposition 4.4. *Let u be a very singular subsolution to* (1.1)*. Then*

$$0 \le u(t,x) \le \Gamma_{p,q}(|x|), \quad (t,x) \in Q_\infty. \tag{4.9}$$

Proof. We adapt the proof of [5, Lemma 3.4]. At a formal level, the result follows from Lemma 2.9 since we can view a very singular solution as having an initial condition satisfiying $R(u_0) = 0$. More precisely, let $r > 0$ and define $D_r := \{x \in \mathbb{R}^N :\ |x| > r\}$. By Lemma 2.8, $S : (t,x) \longmapsto \Gamma_{p,q}(|x| - r)$ is a supersolution to (1.1) in $(0,\infty)\times D_r$ with $u(t,x) < \infty = S(t,x)$ if $(t,x) \in (0,\infty)\times\partial D_r$ and $u(0,x) = 0 \le S(x)$ for $x \in D_r$ by Proposition 4.3. Since u is a subsolution to (1.1) in Q_∞ and thus also in $(0,\infty)\times D_r$, the comparison principle gives $u(t,x) \le \Gamma_{p,q}(|x| - r)$ for any $(t,x) \in (0,\infty)\times D_r$. Fix now $x_0 \in \mathbb{R}^N$, $x_0 \ne 0$. Then $x_0 \in D_r$ for any $r \in (0,|x_0|)$, hence $u(t,x_0) \le \Gamma_{p,q}(|x_0| - r)$, for any $t > 0$ and $r \in (0,|x_0|)$. The conclusion follows by letting $r \to 0$ in the previous inequality. □

We next prove that very singular subsolutions also enjoy the temporal decay estimates (2.8).

Proposition 4.5. *If u is a very singular subsolution to* (1.1) *in Q_∞, the following estimates hold:*

$$t^{\alpha-N\beta}\|u(t)\|_1 + t^{\alpha}\|u(t)\|_\infty \le K_\gamma, \quad t>0, \tag{4.10}$$

where γ and K_γ are defined in (2.17) *and Proposition 2.6, respectively. In addition, if u is a very singular solution to* (1.1) *in Q_∞,*

$$t^{\alpha+\beta}\|\nabla u(t)\|_\infty \le K_\gamma, \quad t>0. \tag{4.11}$$

Proof. At a formal level, since u is a very singular subsolution, its initial condition is somehow concentrated at $x=0$. It thus "vanishes" outside the origin and the conditions on the initial data in Proposition 2.6 are fulfilled. As more regularity on the initial condition is needed to apply this result, we provide a rigorous proof now. Consider $\tau>0$. According to (4.1) and (4.9), $u(\tau)$ satisfies (1.7) and (2.7) with $\kappa=\gamma$ and we infer from Proposition 2.6 that the solution u^τ to (1.1) in $(\tau,\infty)\times\mathbb{R}^N$ with initial condition $u^\tau(\tau)=u(\tau)$ satisfies

$$\begin{aligned}(t-\tau)^{\alpha-N\beta}\|u^\tau(t)\|_1 + (t-\tau)^{\alpha}\|u^\tau(t)\|_\infty \le& K_\gamma,\\ (t-\tau)^{\alpha+\beta}\|\nabla u^\tau(t)\|_\infty \le& K_\gamma,\end{aligned}$$

for $t>\tau$. Now, if u is a very singular subsolution to (1.1), the comparison principle gives $u\le u^\tau$ in $(\tau,\infty)\times\mathbb{R}^N$ and (4.10) follows at once from the previous estimate after letting $\tau\to 0$. Next, if u is a very singular subsolution to (1.1), we obviously have $u^\tau=u$ and thus (4.11). $\square$

Finally, the last preliminary result concerns some local estimates on small balls for very singular subsolutions. It is similar to [5, Lemma 3.6] for $p=2$, and its proof adapts an argument from [16, p. 186-187].

Proposition 4.6. *For $y\in\mathbb{R}^N$ and $\varrho>0$, let $\sigma_{y,\varrho}$ be the solution to*

$$-\Delta_p\sigma_{y,\varrho}=1 \quad in \;\; B_\varrho(y), \qquad \sigma_{y,\varrho}=0 \quad on \;\; \partial B_\varrho(y). \tag{4.12}$$

For every $\lambda\in(0,\infty)$, there exists $A_{\lambda,\varrho}>0$ depending only on N, p, ϱ, and λ such that, if u is a very singular subsolution to (1.1)*, $y\in\mathbb{R}^N\setminus\{0\}$, and $0<\varrho<|y|$, we have*

$$u(t,x)\le\lambda\, e^{A_{\lambda,\varrho}t}\,\exp\left(\frac{1}{\sigma_{y,\varrho}(x)}\right), \quad (t,x)\in(0,\infty)\times B_\varrho(y). \tag{4.13}$$

Proof. We fix $y\in\mathbb{R}^N$, $\varrho\in(0,|y|)$, $\lambda>0$, and define

$$w(t,x):=\lambda\, e^{At}\,\exp\left(\frac{1}{\sigma(x)}\right), \qquad (t,x)\in(0,\infty)\times B_\varrho(y),$$

where $\sigma=\sigma_{y,\varrho}$, the dependence on y and ϱ being omitted for simplicity. We wish to choose $A>0$ such that

$$\partial_t w-\Delta_p w+|\nabla w|^q\ge 0 \quad \text{in } (0,\infty)\times B_\varrho(y). \tag{4.14}$$

To this end, we calculate:

$$\partial_t w(t,x)=A\;w(t,x), \quad \nabla w(t,x)=-\frac{w(t,x)}{\sigma(x)^2}\;\nabla\sigma(x),$$

hence

$$|\nabla w(t,x)|^q = \frac{|\nabla\sigma(x)|^q}{\sigma(x)^{2q}}\ w(t,x)^q$$

and

$$|\nabla w(t,x)|^{p-2}\nabla w(t,x) = -\frac{|\nabla\sigma(x)|^{p-2}}{\sigma(x)^{2(p-1)}}\ w(t,x)^{p-1}\ \nabla\sigma(x).$$

It follows from (4.12) that

$$\begin{aligned}\Delta_p w(t,x) =& 2(p-1)\ \frac{|\nabla\sigma(x)|^p}{\sigma(x)^{2p-1}}\ w(t,x)^{p-1} + (p-1)\ \frac{|\nabla\sigma(x)|^p}{\sigma(x)^{2p}}\ w(t,x)^{p-1} \\ & - \frac{w(t,x)^{p-1}}{\sigma(x)^{2(p-1)}}\ \Delta_p\sigma(x) \\ =& \frac{w(t,x)^{p-1}}{\sigma(x)^{2p}}\ \left[\sigma(x)^2 + (p-1)\ (1+2\sigma(x))\ |\nabla\sigma(x)|^p\right].\end{aligned}$$

Gathering all the previous calculations, we obtain

$$\begin{aligned}\partial_t w - \Delta_p w + |\nabla w|^q =& w^{p-1}\left\{A\ w^{2-p} - \frac{\sigma^2 + (p-1)(1+2\sigma)|\nabla\sigma|^p}{\sigma^{2p}} + \frac{|\nabla\sigma|^q}{\sigma^{2q}}\ w^{q-p+1}\right\} \\ \geq & w^{p-1}\left\{\lambda^{2-p}A\ \exp\left\{\frac{2-p}{\sigma}\right\} - \frac{\left\|\sigma^2 + (1+2\sigma)|\nabla\sigma|^p\right\|_{L^\infty(B_\varrho(y))}}{\sigma^{2p}}\right\}.\end{aligned}$$

Setting $\mu_p := \inf_{r>0}\left\{e^r\ r^{-2p}\right\} > 0$, we end up with

$$\partial_t w - \Delta_p w + |\nabla w|^q \geq \frac{w^{p-1}}{\sigma^{2p}}\left\{\lambda^{2-p}(2-p)^{2p}\mu_p A - \left\|\sigma^2 + (1+2\sigma)|\nabla\sigma|^p\right\|_{L^\infty(B_\varrho(y))}\right\}.$$

Since $\sigma(x) = \varrho^p\ \sigma_{0,1}((x-y)/\varrho)$ for $x \in B_\varrho(y)$, we conclude that (4.14) holds true for a sufficiently large constant $A_{\lambda,\varrho} > 0$ which depends only on N, p, λ, and ϱ.

With this choice, w is a supersolution to (1.1) in $(0,\infty)\times B_\varrho(y)$ which satisfies additionally $w(0,x) \geq 0 = u(0,x)$ for $x \in B_\varrho(y)$ by Proposition 4.3 and $w(t,x) = \infty > u(t,x)$ for $(t,x) \in (0,\infty)\times\partial B_\varrho(y)$ by (4.12). The estimate (4.13) then follows by the comparison principle. □

4.2 The minimal very singular solution

In this section we will construct a special very singular solution and prove that it is minimal among all the very singular solutions and has a self-similar form. As a consequence, it will coincide with the unique radially symmetric self-similar very singular solution obtained in [23], see Theorem 4.2. Recalling the notation u_M for the fundamental solution to (1.1) with mass $M > 0$, we begin with the following preliminary result.

Lemma 4.7. *Let u be a very singular supersolution to* (1.1) *and assume further that*

$$u \in C(Q_\infty) \text{ and } u(t,x) \leq \Gamma_{p,q}(|x|), \qquad (t,x) \in Q_\infty\,.$$

Then, for any $M > 0$, we have $u_M \leq u$ in Q_∞.

Proof. Fix $M > 0$. We borrow ideas from the proofs of [5, Lemma 3.7] and Theorem 1.3 above. As u is a very singular supersolution to (1.1), we have $\|u(t)\|_1 \longrightarrow \infty$ as $t \to 0$ and, for each $k \geq 1$, there exists a non-negative function $u_{0,k} \in L^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ such that

$$\|u_{0,k}\|_1 = M, \quad 0 \leq u_{0,k}(x) \leq u(1/k, x) \leq \Gamma_{p,q}(|x|), \text{ for any } x \in \mathbb{R}^N. \tag{4.15}$$

Denoting the solution to (1.1) with initial condition $u_{0,k}$ by u_k, we argue as in the proof of the existence part of Theorem 1.3 to find a non-negative function $\tilde{u} \in C(Q_\infty)$ and a subsequence of $(u_k)_k$ (not relabeled) with the following properties:

$$\begin{array}{l}\tilde{u} \text{ is a solution to (1.1) in } Q_\infty \text{ and satisfies the estimates (1.12)-(1.13)} \\ \text{with } C_s(M) \text{ and (2.8) with } \kappa = \gamma.\end{array} \tag{4.16}$$

and

$$u_k \longrightarrow \tilde{u} \quad \text{in} \quad C([\tau, T] \times K) \tag{4.17}$$

for all compact subsets K of $\mathbb{R}^N$ and $\tau < t < T$.

It remains to identify the initial condition taken by $\tilde{u}$. On the one hand, since u is a supersolution to (1.1), it readily follows from (4.15) that

$$u_k(t,x) \leq u\left(t + \frac{1}{k}, x\right) \leq \Gamma_{p,q}(|x|)\,, \qquad (t,x) \in Q_\infty\,,$$

whence, owing to (4.17) and the continuity of u in Q_∞,

$$\tilde{u}(t,x) \leq u(t,x) \leq \Gamma_{p,q}(|x|)\,, \qquad (t,x) \in Q_\infty\,. \tag{4.18}$$

On the other hand, consider $\psi \in C_0^\infty(\mathbb{R}^N)$ and $t \in (0,1)$. Owing to (4.15), we may use Proposition 2.7 (a) and deduce that, for all $k \geq 1$,

$$\begin{aligned}\left|\int_{\mathbb{R}^N} (u_k(t,x) - u_{0,k}(x))\,\psi(x)\,dx\right| \leq C(M,1)\ &\Big[\|\psi\|_\infty\ t^{(N+1)(q_*-q)\eta} \\ &+ \|\nabla\psi\|_{p/(2-p)}\ t^{1/p}\Big]\,.\end{aligned} \tag{4.19}$$

It also follows from (4.15) that, for $r > 0$ and $k \geq 1$,

$$\begin{aligned}&\left|\int_{\mathbb{R}^N} u_{0,k}(x)\psi(x)\,dx - M\ \psi(0)\right| = \left|\int_{\mathbb{R}^N} u_{0,k}(x)(\psi(x) - \psi(0))\,dx\right| \\ &\leq 2\|\psi\|_\infty \int_{\{|x|\geq r\}} u(1/k,x)\,dx + \left(\int_{\{|x|\leq r\}} u_{0,k}(x)\,dx\right) \sup_{|x|\leq r}\{|\psi(x) - \psi(0)|\} \\ &\leq 2\|\psi\|_\infty \int_{\{|x|\geq r\}} u(1/k,x)\,dx + M \sup_{|x|\leq r}\{|\psi(x) - \psi(0)|\}.\end{aligned}$$

Combining (4.19) and the above estimate, we obtain, for $k \geq 1$ and $r > 0$,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} \tilde{u}(t,x)\ \psi(x)\,dx - M\ \psi(0)\right| \\
&\leq \left|\int_{\mathbb{R}^N} (\tilde{u}(t,x) - u_k(t,x))\,\psi(x)\,dx\right| + \left|\int_{\mathbb{R}^N} (u_k(t,x) - u_{0,k}(x))\,\psi(x)\,dx\right| \\
&\quad + \left|\int_{\mathbb{R}^N} u_{0,k}(x)\ \psi(x)\,dx - M\psi(0)\right| \\
&\leq \int_{\mathbb{R}^N} |\tilde{u}(t,x) - u_k(t,x)|\,|\psi(x)|\,dx \\
&\quad + C(M,1)\ \left[\|\psi\|_\infty\ t^{(N+1)(q_*-q)\eta} + \|\nabla\psi\|_{p/(2-p)}\ t^{1/p}\right] \\
&\quad + 2\|\psi\|_\infty \int_{\{|x|\geq r\}} u(1/k,x)\,dx + M \sup_{|x|\leq r}\{|\psi(x) - \psi(0)|\}.
\end{aligned}$$

Since $t > 0$, $r > 0$, and ψ is compactly supported, we first let $k \to \infty$ in the above inequality and use (4.3) and (4.17) to conclude that

$$\begin{aligned}
\left|\int_{\mathbb{R}^N} \tilde{u}(t,x)\ \psi(x)\,dx - M\ \psi(0)\right| \leq & C(M,1)\ \left[\|\psi\|_\infty\ t^{(N+1)(q_*-q)\eta} + \|\nabla\psi\|_{p/(2-p)}\ t^{1/p}\right] \\
& + M \sup_{|x|\leq r}\{|\psi(x) - \psi(0)|\}.
\end{aligned}$$

We then let $t \to 0$ and $r \to 0$ and end up with

$$\lim_{t\to 0} \int_{\mathbb{R}^N} \tilde{u}(t,x)\ \psi(x)\,dx = M\ \psi(0) \tag{4.20}$$

for any $\psi \in C_0^\infty(\mathbb{R}^N)$. By a standard density argument, we extend (4.20) to test functions $\psi \in C_0(\mathbb{R}^N)$. In order to extend (4.20) to test functions in $BC(\mathbb{R}^N)$, we proceed as in the proof of the existence part of Theorem 1.3 with the difference that the control for large x is here provided by $\Gamma_{p,q}$ thanks to the upper bound (4.18) and Lemma 2.8. The uniqueness statement of Theorem 1.3 then implies that $\tilde{u} = u_M$. Recalling (4.18) completes the proof. $\square$

The next result shows more properties of the fundamental solutions u_M.

Lemma 4.8. *(a) For each $M > 0$ and $t > 0$, $u_M(t)$ is a radially symmetric function, and $u_{M_1}(t) \leq u_{M_2}(t)$ if $0 < M_1 \leq M_2 < \infty$.*

(b) For each $M > 0$, the function u_M satisfies

$$0 \leq u_M(t,x) \leq \Gamma_{p,q}(|x|), \quad (t,x) \in Q_\infty \tag{4.21}$$

as well as the estimates (1.12)-(1.13) with $C_s(M)$ and (2.8) with $\kappa = \gamma$.

(c) For each $M > 0$ and any $r > 0$, there exist a constant $C(r)$ depending only on r, p, q, and N such that

$$\int_{\{|x|\geq r\}} u_M(t,x)\,dx \leq C(r)\ t^{1/p}\,, \qquad t \in (0,1). \tag{4.22}$$

Proof. The proof of part (a) is identical to the proof of [7, Lemma 3.3] to which we refer. Next, it is easy to see that Proposition 4.3 is also valid for the fundamental solutions u_M and the estimate (4.21) can be proved as Proposition 4.4. We then infer from (3.10) and (4.21) that Propositions 2.5 and 2.6 can be applied to $(t,x) \longmapsto u_M(t+\tau,x)$ for any arbitrary small τ from which the validity of (1.12)-(1.13) with $C_s(M)$ and (2.8) with $\kappa=\gamma$ follows after passing to the limit $\tau \to 0$. Finally, let $\tau > 0$, $r > 0$, and two non-negative functions $\xi \in C_0^\infty(\mathbb{R}^N)$ and $\zeta \in C^\infty(\mathbb{R}^N)$ such that

$$0 \le \xi \le 1\,, \qquad \xi(x) = 1 \quad \text{if} \quad |x| < 1 \quad \text{and} \quad \xi(x) = 0 \quad \text{if} \quad |x| > 2$$

and

$$0 \le \zeta \le 1\,, \qquad \zeta(x) = 0 \quad \text{if} \quad |x| < r/2 \quad \text{and} \quad \zeta(x) = 1 \quad \text{if} \quad |x| > r\,.$$

For $R > 0$ and $x \in \mathbb{R}^N$, we set $\xi_R(x) = \xi(x/R)$. Since $u(\tau)$ satisfies (2.7) with $\kappa = \gamma$ by (4.21) and $\xi_R\zeta \in C_0^\infty(\mathbb{R}^N)$ vanishes in $B_{r/2}(0)$, it follows from (2.10) that, for $t > \tau$ and $R > 0$,

$$\begin{aligned}\int_{\{r<|x|<R\}} u_M(t,x)\,dx \le & \int_{\mathbb{R}^N} u_M(t,x)\ (\xi_R\zeta)(x)\,dx \\ & \le \int_{\mathbb{R}^N} u_M(\tau)\ (\xi_R\zeta)(x)\,dx + C(\gamma,r/2)\ \|\nabla(\xi_R\zeta)\|_{p/(2-p)}\ (t-\tau)^{1/p}\,.\end{aligned}$$

Letting $\tau \to 0$, we find, since $\xi_R\zeta$ vanishes in a neighborhood of $x = 0$,

$$\int_{\{r<|x|<R\}} u_M(t,x)\,dx \le C(\gamma,r/2)\ \|\nabla(\xi_R\zeta)\|_{p/(2-p)}\ t^{1/p}\,.$$

Combining (4.21) with the previous inequality, we obtain

$$\begin{aligned}\int_{\{|x|>r\}} u_M(t,x)\,dx \le & \int_{\{r<|x|<R\}} u_M(t,x)\,dx + \int_{\{|x|>R\}} \Gamma_{p,q}(|x|)\,dx \\ \le & C(\gamma,r/2)\ \|\nabla(\xi_R\zeta)\|_{p/(2-p)}\ t^{1/p} + \int_{\{|x|>R\}} \Gamma_{p,q}(|x|)\,dx \\ \le & C(r)\ \left(\|\nabla(\xi_R\zeta)\|_{p/(2-p)}\ t^{1/p} + R^{-(\alpha-N\beta)/\beta}\right)\,.\end{aligned}$$

Now,

$$\begin{aligned}\|\nabla(\xi_R\zeta)\|_{p/(2-p)} \le & \|\zeta\nabla\xi_R\|_{p/(2-p)} + \|\xi_R\nabla\zeta\|_{p/(2-p)} \\ \le & R^{-(N+1)(p-p_c)/p}\|\nabla\xi\|_{p/(2-p)} + \|\nabla\zeta\|_{p/(2-p)}\,,\end{aligned}$$

and thus

$$\int_{\{|x|>r\}} u_M(t,x)\,dx \le C(r)\ \left(t^{1/p} + R^{-(N+1)(p-p_c)/p}\ t^{1/p} + R^{-(\alpha-N\beta)/\beta}\right)\,.$$

Letting $R \to \infty$ gives (4.22). □

We are now ready to construct the minimal very singular solution. By Lemma 4.8, for any $t > 0$, the sequence $(u_M(t))_{M>0}$ is non-decreasing and uniformly bounded by $\Gamma_{p,q}$. Thus, we can define

$$\overline{U}(t,x) := \sup_{M>0}\{u_M(t,x)\} = \lim_{M\to\infty} u_M(t,x)\,, \qquad (t,x) \in Q_\infty\,. \tag{4.23}$$

Using once more Lemma 4.8, we see that $\overline{U}(t)$ is radially symmetric for any $t > 0$. Moreover, a first outcome of Proposition 4.4 and Lemma 4.7 is that

$$\overline{U} \le u \quad \text{in} \quad Q_\infty \quad \text{for any very singular solution} \quad u \quad \text{to (1.1).} \tag{4.24}$$

It remains to show that $\overline{U}$ is a very singular solution to (1.1).

Proposition 4.9. *The function $\overline{U}$ constructed in* (4.23) *is a very singular solution to* (1.1)*. Moreover, $\overline{U} = U$, the latter being defined in Theorem 4.2.*

Proof. We first prove that $\overline{U}$ has the expected behavior as $t \to 0$. Let $r > 0$. On the one hand, if $M > 0$, we have $\overline{U} \ge u_M$ in Q_∞ by (4.23) and thus

$$\liminf_{t\to 0} \int_{\{|x|\le r\}} \overline{U}(t,x)\,dx \ge \lim_{t\to 0} \int_{\{|x|\le r\}} u_M(t,x)\,dx = M\,,$$

from which the expected concentration (4.2) of $\overline{U}$ at the origin follows. On the other hand, we infer from the monotone convergence theorem and (4.22) that

$$\int_{\{|x|\ge r\}} \overline{U}(t,x)\,dx = \lim_{M\to\infty} \int_{\{|x|\ge r\}} u_M(t,x)\,dx \le C(r)\ t^{1/p}\,.$$

Letting $t \to 0$ gives the expected vanishing (4.3) outside the origin.

Finally, it follows from Lemma 4.8 (b) that $(u_M)_M$ is bounded in $L^\infty(\tau,\infty;W^{1,\infty}(\mathbb{R}^N))$ for any $\tau > 0$. This property and Lemma 2.4 ensure the time equicontinuity of the family $(u_M)_M$ in $(\tau,\infty)\times\mathbb{R}^N$ for all $\tau > 0$. We then deduce from the Arzelà-Ascoli theorem that $(u_M)_M$ is relatively compact in $C([\tau,T]\times K)$ for all compact subsets K of $\mathbb{R}^N$ and $0 < \tau < T$. Recalling (4.23), we conclude that $(u_M)_M$ converges to $\overline{U}$ uniformly in compact subsets of Q_∞. Consequently, thanks to the stability of viscosity solutions [30, Theorem 6.1], $\overline{U}$ is a viscosity solution to (1.1), and thus a very singular solution in the sense of Definition 4.1.

It remains to prove that $\overline{U}$ has a self-similar form which follows from the scale invariance of (1.1) and is now a standard step. Indeed, for $\lambda \in (0,\infty)$ and $M \in (0,\infty)$, define a rescaled version of u_M by

$$u_M^\lambda(t,x) := \lambda^{(p-q)/(2q-p)} u_M(\lambda t, \lambda^{(q-p+1)/(2q-p)} x)\,, \qquad (t,x) \in Q_\infty. \tag{4.25}$$

By straightforward calculations, we find that u_M^λ is a solution to (1.1). To identify its initial trace, we consider $\psi \in BC(\mathbb{R}^N)$ and write

$$\begin{aligned}\int_{\mathbb{R}^N} u_M^\lambda(t,x)\ \psi(x)\,dx &= \lambda^{(p-q)/(2q-p)} \int_{\mathbb{R}^N} u_M(\lambda t, \lambda^{(q-p+1)/(2q-p)}x)\ \psi(x)\,dx \\ &= \lambda^{(N+1)(q_*-q)/(2q-p)} \int_{\mathbb{R}^N} u_M(\lambda t, y)\ \psi(\lambda^{-(q-p+1)/(2q-p)}y)\,dy.\end{aligned}$$

Letting $t \to 0$, we find that the initial condition of u_M^λ is $\lambda^{(N+1)(q_*-q)/(2q-p)} M\delta_0$. By Theorem 1.3, we obtain $u_M^\lambda = u_{\lambda^{(N+1)(q_*-q)/(2q-p)}M}$. We now pass to the limit as $M \to \infty$ and deduce from (4.23) that

$$\overline{U}(t,x) = \lambda^{(p-q)/(2q-p)}\ \overline{U}(\lambda t, \lambda^{(q-p+1)/(2q-p)}x), \quad (t,x) \in Q_\infty\,.$$

Therefore, $\overline{U}$ has a self-similar form and since it is obviously radially symmetric due to Lemma 4.8 (a) and (4.23), we infer from Theorem 4.2 that $\overline{U} = U$. □

A further outcome of the above analysis is the following result which is a straightforward consequence of Lemma 4.7, (4.23), and Proposition 4.9.

Corollary 4.10. *If u is a very singular supersolution to* (1.1) *in Q_∞ such that*

$$u \in C(Q_\infty) \text{ and } u(t,x) \le \Gamma_{p,q}(|x|), \qquad (t,x) \in Q_\infty\,,$$

then

$$U(t,x) \le u(t,x), \qquad (t,x) \in Q_\infty.$$

4.3 The maximal very singular solution

We begin with the following general result for very singular subsolutions to (1.1).

Proposition 4.11. *Let u be a very singular subsolution to* (1.1)*. Then there exists a very singular solution $\overline{u}$ such that $u \le \overline{u}$ in Q_∞.*

Proof. Fix $\tau > 0$ and let u^τ be the solution to (1.1) in $(\tau,\infty)\times\mathbb{R}^N$ with initial condition $u^\tau(\tau) = u(\tau)$. The comparison principle and Proposition 4.4 then ensure that

$$u(t,x) \le u^\tau(t,x) \le \Gamma_{p,q}(|x|)\,, \qquad (t,x) \in (\tau,\infty)\times\mathbb{R}^N\,. \tag{4.26}$$

Moreover, the function $u(\tau)$ satisfies (1.7) and (2.7) (with $\kappa=\gamma$) by Proposition 4.4 and it follows from Proposition 2.6 that, for $t>\tau$,

$$(t-\tau)^{\alpha-N\beta}\|u^\tau(t)\|_1 + (t-\tau)^\alpha\|u^\tau\|_\infty \le K_\gamma, \tag{4.27}$$

and

$$(t-\tau)^{\alpha+\beta}\|\nabla u^\tau(t)\|_\infty \le K_\gamma. \tag{4.28}$$

We also notice that, if $0<\tau_1<\tau_2$, the inequality (4.26) implies that $u^{\tau_2}(\tau_2) = u(\tau_2) \le u^{\tau_1}(\tau_2)$, whence

$$u^{\tau_1}(t,x) \ge u^{\tau_2}(t,x)\,, \qquad (t,x) \in (\tau_2,\infty)\times\mathbb{R}^N\,, \tag{4.29}$$

by the comparison principle. Owing to (4.26), (4.27), and (4.29), we may define the pointwise limit

$$W(t,x) := \sup_{\tau\in(0,t/2)}\{u^\tau(t,x)\} = \lim_{\tau\to 0} u^\tau(t,x), \quad (t,x)\in Q_\infty. \tag{4.30}$$

The remainder of the proof is devoted to proving that W is a very singular solution to (1.1) in Q_∞. Consider $n\ge 1$. By (4.27) and (4.28), the family $\{u^\tau\ :\ \tau\in(0,1/2n)\}$ is bounded in $L^\infty(1/n,n;W^{1,\infty}(\mathbb{R}^N))$ which allows us to apply Lemma 2.4 and deduce from the Arzelà-Ascoli theorem that $\{u^\tau\ :\ \tau\in(0,1/2n)\}$ is relatively compact in $C((1/n,n)\times B_n(0))$. Consequently, the pointwise convergence (4.30) of $(u^\tau)_\tau$ to W can be improved to convergence in $C((1/n,n)\times B_n(0))$ for all $n\ge 1$, from which we deduce that W is a viscosity solution to (1.1) in Q_∞ by the stability of viscosity solutions [30, Theorem 6.1]. We may also use this convergence to pass to the limit as $\tau\to 0$ in (4.26) and obtain

$$u(t,x) \le W(t,x) \le \Gamma_{p,q}(|x|)\,, \qquad (t,x)\in Q_\infty\,. \tag{4.31}$$

It remains to prove that the function W has the expected behavior as $t\to 0$. Since u is a very singular subsolution to (1.1), it satisfies (4.2) and so does W by (4.31). The study of the behavior of W outside the origin requires more work. Let $\zeta \in C^\infty(\mathbb{R}^N)$ be such that $0 \le \zeta \le 1$,

$$\zeta(x) = 1 \text{ if } |x| \ge 1, \quad \zeta(x) = 0 \text{ if } |x| \le \frac{1}{2}.$$

Fix $r > 0$ and define $\zeta_r(x) = \zeta(x/r)$ for $x \in \mathbb{R}^N$. It follows from (2.1) that, for $t > 0$ and $\tau \in (0, t/2)$,

$$\begin{aligned}\int_{\mathbb{R}^N} u^\tau(t,x)\ \zeta_r(x)\,dx \le& \int_{\mathbb{R}^N} u^\tau(\tau,x)\ \zeta_r(x)\,dx + \int_\tau^t \int_{\mathbb{R}^N} |\nabla u^\tau(s,x)|^{p-1}\ |\nabla\zeta_r(x)|\,dx\,ds \\ \le& \int_{\{|x|\ge r/2\}} u(\tau,x)\,dx \\ &+ \int_\tau^t \int_{\{r/2<|x|<r\}} |\nabla u^\tau(s,x)|^{p-1}\ |\nabla\zeta_r(x)|\,dx\,ds,\end{aligned} \tag{4.32}$$

since $u^\tau(\tau) = u(\tau)$ by definition. On the one hand, Fatou's lemma and (4.30) give

$$\int_{\mathbb{R}^N} W(t,x)\ \zeta_r(x)\ dx \le \liminf_{\tau\to 0} \int_{\mathbb{R}^N} u^\tau(t,x)\ \zeta_r(x)\ dx\,. \tag{4.33}$$

On the other hand, since u is a very singular subsolution, we have

$$\lim_{\tau\to 0} \int_{\{|x|\ge r/2\}} u(\tau,x)\,dx = 0\,, \tag{4.34}$$

while (2.3), (4.26), and (4.27) give, for $s > \tau$,

$$\begin{aligned}|\nabla u^\tau(s,x)|^{p-1} \le& C\left(\left\|u^\tau\left(\frac{s+\tau}{2}\right)\right\|_\infty^{1/\alpha p} + (s-\tau)^{-1/p}\right)^{p-1}\ (u^\tau(s,x))^{2(p-1)/p} \\ \le& C\ (s-\tau)^{-(p-1)/p}\ \Gamma_{p,q}(|x|)^{2(p-1)/p}\,.\end{aligned}$$

Thus

$$\int_\tau^t \int_{\{r/2<|x|<r\}} |\nabla u^\tau(s,x)|^{p-1}\ |\nabla\zeta_r(x)|\,dx\,ds \le C(r)\ t^{1/p}\|\nabla\zeta\|_\infty. \tag{4.35}$$

Combining (4.32), (4.33), (4.34), and (4.35) leads us to

$$\int_{\mathbb{R}^N} W(t,x)\ \zeta_r(x)\,dx \le C(r)\ t^{1/p}\ \|\nabla\zeta\|_\infty.$$

Using the properties of ζ and letting $t \to 0$ in the above inequality, we conclude that W satisfies (4.3). Summarizing, we have established that W is a very singular solution to (1.1) in Q_∞ which lies above u by (4.31). □

We are now ready to construct the maximal very singular solution to (1.1). We denote the set of very singular solutions to (1.1) in Q_∞ by $\mathcal{S}$. Since $U \in \mathcal{S}$, $\mathcal{S}$ is non-empty and we may define

$$V(t,x) := \sup_{u\in\mathcal{S}}\{u(t,x)\}, \quad (t,x) \in Q_\infty. \tag{4.36}$$

We prove next that V is itself a very singular solution to (1.1). We begin with the following bounds.

Lemma 4.12. *For $t > 0$, we have*

$$t^\alpha \ \|V(t)\|_\infty + t^{\alpha+\beta}\|\nabla V(t)\|_\infty \le K_\gamma, \tag{4.37}$$

and

$$U(t,x) \le V(t,x) \le \Gamma_{p,q}(|x|), \quad x \in \mathbb{R}^N. \tag{4.38}$$

Proof. Since $U \in \mathcal{S}$, the inequality (4.38) follows at once from (4.36) and Proposition 4.4. We next deduce from (4.10) and (4.36) that $\|V(t)\|_\infty \le K_\gamma \ t^{-\alpha}$ for $t > 0$ while (4.11) and (4.36) entail that, for any $x \in \mathbb{R}^N$, $y \in \mathbb{R}^N$, $u \in \mathcal{S}$, and $t > 0$,

$$u(t,x) \le u(t,y) + K_\gamma \ t^{-(\alpha+\beta)} \ |x-y| \le V(t,y) + K_\gamma \ t^{-(\alpha+\beta)} \ |x-y|.$$

Hence, passing to the supremum over $u \in \mathcal{S}$

$$V(t,x) \le V(t,y) + K_\gamma \ t^{-(\alpha+\beta)} \ |x-y|,$$

and $V(t)$ is Lipschitz continuous for all $t > 0$ with Lipschitz constant $K_\gamma \ t^{-(\alpha+\beta)}$. Consequently, $V(t) \in W^{1,\infty}(\mathbb{R}^N)$ and satisfies (4.37). □

We can now establish the main property of V.

Lemma 4.13. *V is a very singular subsolution to* (1.1)*.*

Proof. Since V is the supremum of a family of viscosity solutions to (1.1) by (4.36), the fact that V is a viscosity subsolution to (1.1) follows from [1, Proposition V.2.11]. The regularity $V(t) \in L^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ for $t > 0$ is a consequence of Lemma 4.12 and the integrability at infinity of $\Gamma_{p,q}$ (see Lemma 2.8). Also, the concentrating property (4.2) at the origin as $t \to 0$ follows at once from (4.36) since $U \in \mathcal{S}$. It remains to check that $V(t)$ vanishes outside the origin as $t \to 0$. For that purpose, let $r > 0$ and $R > r$. Since the annulus $K(r,R) := \{x \in \mathbb{R}^N \ : \ r/2 \le |x| \le R\}$ is compact, there is a finite number l of points $(y_i)_{1 \le i \le l}$ in $\mathbb{R}^N$ such that

$$K(r,R) \subset \bigcup_{i=1}^{l} B_{r/8}(y_i). \tag{4.39}$$

We infer from (4.13) that, for any $1 \le i \le l$, $\lambda > 0$, $t > 0$, and $u \in \mathcal{S}$, we have

$$u(t,x) \le \lambda e^{A_{\lambda,r/4} t} \ \exp\left(\frac{1}{\sigma_{y_i,r/4}(x)} \right), \quad x \in B_{r/8}(y_i).$$

The above estimate being valid for all $u \in \mathcal{S}$ we conclude that, for any $1 \le i \le l$, $\lambda > 0$, and $t > 0$,

$$V(t,x) \le \lambda e^{A_{\lambda,r/4} t} \ \exp\left(\frac{1}{\sigma_{y_i,r/4}(x)} \right), \quad x \in B_{r/8}(y_i). \tag{4.40}$$

Recalling (4.39), we infer from (4.38) and (4.40) that, for $t > 0$ and $\lambda > 0$,

$$\begin{aligned} \int_{\{|x|\ge r\}} V(t,x)\,dx &= \int_{K(r,R)} V(t,x)\,dx + \int_{\{|x|>R\}} V(t,x)\,dx \\ &\le \lambda \ e^{A_{\lambda,r/4} t} \ \sum_{i=1}^{l} \int_{B_{r/8}(y_i)} \exp\left(\frac{1}{\sigma_{y_i,r/4}(x)} \right) dx + \int_{\{|x|>R\}} \Gamma_{p,q}(|x|)\,dx. \end{aligned}$$

Passing first to the limit $t \to 0$ and then $\lambda \to 0$ gives

$$\limsup_{t\to 0} \int_{\{|x|\ge r\}} V(t,x)\,dx \le \int_{\{|x|>R\}} \Gamma_{p,q}(|x|)\,dx$$

for all $R > r$. Thanks to Lemma 2.8, the right-hand side of the above inequality converges to zero as $R \to \infty$, so that V satisfies (4.3) and the proof is complete. $\square$

We are now in a position to identify V.

Proposition 4.14. *The function V defined in* (4.36) *is a very singular solution in the sense of Definition 4.1. Moreover, it is radially symmetric and has self-similar form, thus coinciding with the unique self-similar very singular solution U given by Theorem 4.2.*

A straightforward consequence of Proposition 4.14 is that $\mathcal{S} = \{U\}$, which proves Theorem 1.2.

Proof. It follows from Proposition 4.11 and Lemma 4.13 that there exists a very singular solution $\overline{u}$ to (1.1) such that

$$V(t,x) \le \overline{u}(t,x) \quad \text{for all } (t,x) \in (0,\infty)\times\mathbb{R}^N.$$

The definition (4.36) of V implies that $V \equiv \overline{u}$ and thus V is the maximal very singular solution. The radial symmetry and self-similarity of V then follow from the scaling and rotational invariances of (1.1). $\square$

4.4 A comparison principle

An interesting consequence of the uniqueness of the very singular solutions to (1.1) is the following comparison principle for the related elliptic equation

$$-\Delta_p v + |\nabla v|^q - \alpha v - \beta y\cdot\nabla v = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{4.41}$$

Theorem 4.15. *Let v_1 be a viscosity subsolution and v_2 be a viscosity supersolution to* (4.41) *in $\mathbb{R}^N$, such that*

$$v_i \in L^1(\mathbb{R}^N)\cap W^{1,\infty}(\mathbb{R}^N), \quad v_i \ge 0, \quad v_i \not\equiv 0 \quad i=1,2. \tag{4.42}$$

Assume that

$$\lim_{R\to\infty} \int_{\{|y|\ge R\}} v_i(y)|y|^{\alpha/\beta-N}\,dy = 0,\ i=1,2, \quad \text{and} \quad v_2(y) \le \Gamma_{p,q}(|y|), \quad y\in\mathbb{R}^N. \tag{4.43}$$

Then $v_1(y) \le f_U(y) \le v_2(y)$ for all $y \in \mathbb{R}^N$, where f_U is the profile of the very singular solution U to (1.1), *see Theorem 4.2.*

Besides its interest in itself, this comparison principle will also be useful to settle the asymptotic behavior in Section 5.

Proof. For $i = 1,2$, define

$$u_i(t,x) := t^{-\alpha} v_i(xt^{-\beta}), \quad (t,x) \in Q_\infty.$$

It is then straightforward to check that u_1 is a subsolution and u_2 is a supersolution to (1.1) in Q_∞. Moreover, we have

$$u_i \in C(Q_\infty) \quad \text{and} \quad u_i(t) \in L^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N), \quad t > 0\,, \quad i = 1, 2.$$

On the one hand, for $i = 1, 2$ and any $r > 0$, we have

$$\begin{aligned}\int_{\{|x|\geq r\}} u_i(t,x)\,dx &= t^{N\beta-\alpha} \int_{\{|y|\geq rt^{-\beta}\}} v_i(y)|y|^{\alpha/\beta-N}|y|^{N-\alpha/\beta}\,dy \\ &\leq r^{N-\alpha/\beta} \int_{\{|y|\geq rt^{-\beta}\}} v_i(y)|y|^{\alpha/\beta-N}\,dy,\end{aligned}$$

which tends to 0 as $t \to 0$ by (4.43). On the other hand, since $v_i \not\equiv 0$, there is $r_0 > 0$ sufficiently large such that

$$\int_{B_{r_0}(0)} v_i(y)\,dy > 0\,, \quad i = 1, 2\,.$$

Consequently, for $t > 0$ sufficiently small ($t \in \big(0, (r/r_0)^{1/\beta}\big)$), we have

$$\int_{\{|x|\leq r\}} u_i(t,x)\,dx = t^{N\beta-\alpha} \int_{\{|y|\leq rt^{-\beta}\}} v_i(y)\,dy > t^{N\beta-\alpha} \int_{B_{r_0}(0)} v_i(y)\,dy,$$

which tends to $+\infty$ as $t \to 0$, since $N\beta - \alpha < 0$. It follows that u_1 is a very singular subsolution to (1.1) and u_2 is a very singular supersolution to (1.1). Furthermore

$$u_2(t,x) = t^{-\alpha} v_2(xt^{-\beta}) \leq \gamma|x|^{-\alpha/\beta} = \Gamma_{p,q}(|x|),$$

for any $(t,x) \in Q_\infty$. By Theorem 1.2, Corollary 4.10, and Proposition 4.11, we obtain

$$u_1 \leq U \leq u_2 \quad \text{in} \quad Q_\infty.$$

We reach the conclusion by going back to the original variables. $\square$

5 Convergence to self-similarity

With all the preparations done in the previous sections, we are now ready to prove the main result about asymptotic convergence. The proof will be divided into several steps.

Proof of Theorem 1.1. Let us first notice that the condition (1.8) implies that $R(u_0) < \infty$, where $R(u_0)$ is defined in (2.18). Moreover, there exists a sufficiently large constant $\kappa > 0$ such that

$$u_0(x) \leq \kappa|x|^{-\alpha/\beta} \quad \text{for any} \quad x \in \mathbb{R}^N.$$

Step 1. Self-similar variables. In a first step, we pass to self-similar variables and define the new variables (s, y) and function v by

$$u(t,x) =: (1+t)^{-\alpha} v(s,y), \quad s := \ln(1+t),\ y := x(1+t)^{-\beta}. \tag{5.1}$$

Then v solves the equation

$$\partial_s v - \Delta_p v + |\nabla v|^q - \alpha v - \beta y \cdot \nabla v = 0, \quad (s,y) \in Q_\infty, \tag{5.2}$$

with initial condition $v(0) = u_0$ in $\mathbb{R}^N$.

Step 2. Estimates for v. Starting from the estimates established for u, we can deduce estimates for v in similar norms as follows. First, recalling the homogeneity of $\Gamma_{p,q}$, we deduce from (2.19) that

$$\begin{aligned} v(s,y) = e^{\alpha s} u(e^s - 1, y e^{\beta s}) &\le e^{\alpha s} \Gamma_{p,q}(|y| e^{\beta s} - R(u_0)) \\ &\le \Gamma_{p,q}(|y| - R(u_0) e^{-\beta s}) \end{aligned} \tag{5.3}$$

for any $(s,y) \in Q_\infty$. Then, the estimates (2.8) can be easily transformed into the following ones for v:

$$\begin{aligned} \|v(s)\|_1 + \|v(s)\|_\infty + \|\nabla v(s)\|_\infty &\le \left[\left(\frac{e^s}{e^s - 1} \right)^{\alpha - N\beta} + \left(\frac{e^s}{e^s - 1} \right)^{\alpha} + \left(\frac{e^s}{e^s - 1} \right)^{\alpha + \beta} \right] K_\kappa \\ &\le 6 K_\kappa, \end{aligned} \tag{5.4}$$

for any $s > \ln 2 > 0$, where K_κ is the constant in (2.8). Finally, the pointwise upper bound (2.23) reads

$$|y|^{\alpha/\beta} v(s,y) \le C \left[\sup_{|z| \ge |y| e^{\beta s}/4} \left\{ u_0(z) |z|^{\alpha/\beta} \right\} + |y|^{-1/\beta} \right] \tag{5.5}$$

for $(s,y) \in (0,\infty) \times (\mathbb{R}^N \setminus \{0\})$.

Step 3. Lower bound for v. We infer from [22, Proposition 1.8] that $u(t,x) > 0$ for $(t,x) \in Q_\infty$. In particular, $u(1,0) > 0$ and, since $u(1,\cdot) \in C(\mathbb{R}^N)$, there is $m_0 > 0$ such that

$$u(1,x) \ge m_0\,, \qquad x \in B_1(0)\,. \tag{5.6}$$

Next, according to the analysis performed in [23] (in particular, Lemma 2.1, Lemma 2.8, Lemma 2.10, Proposition 2.11, and Proposition 2.16 therein), there exists $a_* > 0$ such that, for $a \in (0, a_*)$, the maximal solution g_a defined on $[0, R_m(a))$ to the Cauchy problem

$$\left\{ \begin{array}{l} (|g_a'|^{p-2} g_a')'(r) + \dfrac{N-1}{r} (|g_a'|^{p-2} g_a')(r) + \alpha g_a(r) + \beta r g_a'(r) - |g_a'(r)|^q = 0, \\ \\ g_a(0) = a,\ g_a'(0) = 0, \end{array} \right. \tag{5.7}$$

has the following properties: there is $R(a) \in (0, R_m(a))$ such that

$$0 < g_a(r) \le a \quad \text{for} \quad r \in [0, R(a))\,, \quad g_a(R(a)) = 0\,, \quad \text{and} \quad g_a'(R(a)) < 0\,. \tag{5.8}$$

Introducing

$$G_{a,\lambda}(t,x) := \left\{ \begin{array}{ll} \lambda^{p/(2-p)}\, t^{-\alpha}\, g_a\left(\lambda |x| t^{-\beta}\right) & \text{if} \quad |x| \in \left[0, R(a) t^\beta/\lambda\right]\,, \\ 0 & \text{if} \quad |x| \ge R(a) t^\beta/\lambda\,, \end{array} \right.$$

for $(a,\lambda) \in (0,a_*) \times (0,1)$, the properties of g_a guarantee that $G_{a,\lambda} \in C(Q_\infty)$ and is a subsolution to (1.1) in Q_∞ (it can be interpreted locally as the maximum of two subsolutions to (1.1) in Q_∞, namely the zero function and $(t,x) \longmapsto \lambda^{p/(2-p)}\, t^{-\alpha}\, g_a\left(\lambda |x| t^{-\beta}\right)$).
Now, we set

$$a_0 := \frac{a_*}{2} \in (0, a_*)\,, \quad t_0 := \frac{1}{R(a_0)^p} \left(\frac{m_0}{a_0} \right)^{2-p}\,, \quad \lambda_0 := R(a_0)\, t_0^\beta\,,$$

and observe that, if $x \in B_{R(a_0)t_0^\beta/\lambda_0}(0) = B_1(0)$, then (5.6) implies that

$$G_{a_0,\lambda_0}(t_0,x) \le a_0 \lambda_0^{p/(2-p)} t_0^{-\alpha} = a_0 \left(\lambda_0 t_0^{-\beta}\right)^{p/(2-p)} t_0^{1/(2-p)} = m_0 \le u(1,x) .$$

Since $u(1,x) > 0 = G_{a_0,\lambda_0}(t_0,x)$ if $x \notin B_{R(a_0)t_0^\beta/\lambda_0}(0)$, we have $u(1,x) \ge G_{a_0,\lambda_0}(t_0,x)$ for all $x \in \mathbb{R}^N$ and the comparison principle entails that

$$u(t+1,x) \ge G_{a_0,\lambda_0}(t+t_0,x) , \qquad (t,x) \in Q_\infty .$$

In particular, for $t > 0$ and $x \in B_{R(a_0)(t+t_0)^\beta/\lambda_0}(0)$,

$$u(t+1,x) \ge \lambda_0^{p/(2-p)} (t+t_0)^{-\alpha} g_{a_0}\left(\lambda_0 |x| (t+t_0)^{-\beta}\right) .$$

In terms of v, the previous lower bound reads

$$v(s,y) \ge \lambda_0^{p/(2-p)} \left(\frac{e^s}{e^s - 2 + t_0}\right)^\alpha g_{a_0}\left(\lambda_0 |y| \left(\frac{e^s}{e^s - 2 + t_0}\right)^\beta\right) , \tag{5.9}$$

for $s > \ln 2$ and $|y| \le (R(a_0)/\lambda_0)((e^s - 2 + t_0)e^{-s})^\beta$.

Step 4. Half-relaxed limits. To complete the proof of the convergence, we introduce the half-relaxed limits [2], in a similar way as it has been previously used in papers on large-time behavior, see [24, 32] for instance. We thus define

$$\tilde{w}_*(s,y) := \liminf_{(\sigma,z,\varepsilon)\to(s,y,0)} v\left(\frac{\sigma}{\varepsilon}, z\right) , \quad \tilde{w}^*(s,y) := \limsup_{(\sigma,z,\varepsilon)\to(s,y,0)} v\left(\frac{\sigma}{\varepsilon}, z\right)$$

for $(s,y) \in Q_\infty$. It is a standard fact that $\tilde{w}_*$ and $\tilde{w}^*$ do not depend on $s > 0$, so that we can define

$$w_*(y) := \tilde{w}_*(1,y) = \tilde{w}_*(s,y), \quad w^*(y) := \tilde{w}^*(1,y) = \tilde{w}^*(s,y), \quad s > 0 .$$

In addition, it follows from [2, Théorème 4.1] that w_* is a viscosity supersolution and w^* is a viscosity subsolution to the stationary equation associated to (5.2), that is, the elliptic equation (4.41). Moreover, the definition of w_* and w^* and (5.9) ensure that

$$w_* \le w^* \quad \text{in} \quad \mathbb{R}^N \quad \text{and} \quad \lambda_0^{p/(2-p)} g_{a_0}(\lambda_0 |y|) \le w_*(y) \quad \text{for} \quad y \in B_{R(a_0)/\lambda_0}(0) . \tag{5.10}$$

An obvious consequence of (5.8) and (5.10) is that w_* and w^* are both not identically equal to zero.

Our aim now is to show that $w_* \equiv w^*$ with the help of Theorem 4.15. In order to apply it, we translate the estimates for v in Step 2 above into estimates for w_* and w^*. We readily notice that (5.3) implies

$$w_*(y) \le w^*(y) \le \Gamma_{p,q}(|y|), \quad \text{for any } y \in \mathbb{R}^N, \tag{5.11}$$

and that (5.4) implies that

$$w_*(y) \le w^*(y) \le 6K_\kappa, \quad \|\nabla w_*\| \le 6K_\kappa, \ \|\nabla w^*\| \le 6K_\kappa, \quad y \in \mathbb{R}^N, \tag{5.12}$$

whence w_* and w^* belong to the space $L^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$. In addition, taking into account the condition (1.8) on u_0, we deduce from (5.5) that

$$w_*(y) \le w^*(y) \le C|y|^{-(\alpha+1)/\beta}, \quad y \in \mathbb{R}^N .$$

Consequently,

$$\int_{\{|y|\ge r\}} (w_*(y) + w^*(y))\, |y|^{\alpha/\beta - N}\, dy \le C \int_r^\infty s^{-1/\beta-1}\, ds = Cr^{-1/\beta}, \tag{5.13}$$

which converges to 0 as $r \to \infty$. Gathering (5.10), (5.11), (5.12), and (5.13), we are in a position to apply Theorem 4.15 and conclude that $w^* \le f_U \le w_*$ in $\mathbb{R}^N$.

Recalling (5.10), we have established that $w_* \equiv w^* = f_U$, which in turn implies that

$$\lim_{\varepsilon\to 0} \sup_{y\in K} \left\{ \left| v\left(\frac{1}{\varepsilon}, y\right) - f_U(y) \right| \right\} = 0$$

for any compact subset K of $\mathbb{R}^N$ by [1, Lemma V.1.9] or [2, Lemme 4.1]. Owing to (5.3) and the decay of f_U as $|x| \to \infty$ (see Theorem 4.2), the above convergence can be improved to the convergence of $v(s)$ to f_U in $L^\infty(\mathbb{R}^N)$ as $s \to \infty$. Going back to the original variables gives (1.9) and ends the proof. $\square$

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